UNIVERSITY OF JOHANNESBURG

Applied Mathematics 3B

Assignment #3 **7:30, 16 August 2011**

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1. Consider the observable

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate the expectation value for a system described by the pure state $(1,0)^T$. Calculate the expectation value for a system described by the mixed state

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

What can we conclude?

2. Consider the observable

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate the maximum expectation value M for a system described by a pure state $\mathbf{x} \in \mathbb{C}^2$ and the pure state(s) $\mathbf{x}$ at which the maximum is achieved. Let

$$\rho := p\mathbf{x}\mathbf{x}^* + (1-p)\mathbf{y}\mathbf{y}^*$$

where $p \in (0,1)$ and $\mathbf{y} \in \mathbb{C}^2$ is linearly independent of $\mathbf{x}$ determined above. Show that the expectation value for a system described by the mixed state ρ is less than M. Can this result be extended to all mixed states?

3. Consider the Hilbert space $\mathbb{C}^4$. Show that the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

describes an observable. Describe the measurement outcomes and associated probabilities when observing (performing the measurement on) a system described by the state

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

What is the state after the measurement?

4. Let A and B be two commuting $n \times n$ Hermitian matrices over $\mathbb{C}$ each with distinct eigenvalues. Thus the corresponding eigenspaces are one-dimensional. Show that every eigenvector $\mathbf{x}$ of A is also an eigenvector of B.

Hint: First show that $B\mathbf{x}$ is either $\mathbf{0}$ or an eigenvector of A.

5. Find the constraints on pure states $\mathbf{x}, \mathbf{y} \in \mathbb{C}^2$ for which, when classically mixed in equal proportions, we find the density matrix

$$\frac{1}{2}\mathbf{x}\mathbf{x}^* + \frac{1}{2}\mathbf{y}\mathbf{y}^* = \frac{1}{2}I_2.$$
