Applied Mathematics 3B

Assignment #2 **Solution**

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1. We have

$$\lambda_1 = 2\hbar\omega, \quad \mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \lambda_2 = 0, \quad \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Thus we find

$$\begin{aligned} e^{-it\lambda_1/\hbar}\mathbf{x}_1\mathbf{x}_1^* + e^{-it\lambda_2/\hbar}\mathbf{x}_2\mathbf{x}_2^* &= \frac{e^{-2it\omega}}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \\ &= e^{-it\omega}\begin{pmatrix} \dfrac{e^{-it\omega}+e^{it\omega}}{2} & i\dfrac{-e^{-it\omega}+e^{it\omega}}{2} \\ i\dfrac{e^{-it\omega}-e^{it\omega}}{2} & \dfrac{e^{-it\omega}+e^{it\omega}}{2} \end{pmatrix} \\ &= e^{-it\omega}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}. \end{aligned}$$

2. The solution is given by

$$\boldsymbol{\psi}(t) = e^{-i\left[\hbar\omega\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\right]t/\hbar}\boldsymbol{\psi}(0) = e^{-i\omega t\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}}\boldsymbol{\psi}(0).$$

Let

$$A := -i\omega t\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

We have

$$e^A = \sum_{j=0}^{\infty}\frac{A^j}{j!}$$

where

$$A^0 = I_2, \qquad A^1 = A, \qquad A^2 = (-2i\omega t)A, \qquad A^3 = AA^2 = (-2i\omega t)A^2 = (-2i\omega t)^2A.$$

Suppose $A^k = (-2i\omega t)^{k-1}A$ (which is clearly true for $k=1$ and $k=2$). It follows that

$$A^{k+1} = AA^k = (-2i\omega t)^{k-1}A^2 = (-2i\omega t)^kA.$$

Thus $A^k = (-2i\omega t)^{k-1}A$ for $k = 1, 2, \ldots$ by induction. We have

$$\begin{aligned} e^A &= \sum_{j=0}^{\infty}\frac{A^j}{j!} = I_2 + \sum_{k=1}^{\infty}\frac{A^k}{k!} \\ &= I_2 + \sum_{k=1}^{\infty}\frac{(-2i\omega t)^{k-1}}{k!}A \\ &= I_2 - \frac{1}{2i\omega t}\sum_{k=1}^{\infty}\frac{(-2i\omega t)^k}{k!}A \\ &= I_2 - \frac{1}{2i\omega t}\left(e^{-2i\omega t}-1\right)A \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2i\omega t}\left(e^{-2i\omega t}-1\right)(-i\omega t)\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\left(e^{-2i\omega t}-1\right)\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} \frac{e^{-2i\omega t}+1}{2} & -i\frac{e^{-2i\omega t}-1}{2} \\ i\frac{e^{-2i\omega t}-1}{2} & \frac{e^{-2i\omega t}+1}{2} \end{pmatrix} \\ &= e^{-it\omega}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}. \end{aligned}$$

Consequently

$$\boldsymbol{\psi}(t) = e^{-it\omega}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}\boldsymbol{\psi}(0).$$

3. Let

$$A = \begin{pmatrix} a & b \\ c & b \end{pmatrix}$$

and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $a, b, c, d, x_1, x_2 \in \mathbb{C}$. Then

$$\mathbf{x}^* A\mathbf{x} = (\,\overline{x_1} \quad \overline{x_2}\,)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a|x_1|^2 + d|x_2|^2 + b\overline{x_1}x_2 + cx_1\overline{x_2}.$$

We also have

$$\begin{aligned} \mathrm{tr}(A\mathbf{x}\mathbf{x}^*) &= \mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(\,\overline{x_1} \quad \overline{x_2}\,)\right) \\ &= \mathrm{tr}\left(\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}(\,\overline{x_1} \quad \overline{x_2}\,)\right) \\ &= \mathrm{tr}\begin{pmatrix} a|x_1|^2 + b\overline{x_1}x_2 & ax_1\overline{x_2} + b|x_2|^2 \\ c|x_1|^2 + d\overline{x_1}x_2 & cx_1\overline{x_2} + d|x_2|^2 \end{pmatrix} \\ &= a|x_1|^2 + b\overline{x_1}x_2 + cx_1\overline{x_2} + d|x_2|^2 \\ &= \mathbf{x}^* A\mathbf{x} \end{aligned}$$

Let A be $n \times n$, and $x \in \mathbb{C}^n$, i.e. $\mathbf{x}$ is $n \times 1$. In the above $n = 2$. Thus $\mathbf{x}^*$ is $1 \times n$. It follows that $\mathrm{tr}(A\mathbf{x}\mathbf{x}^*) = \mathrm{tr}(\mathbf{x}^* A\mathbf{x})$. Since $\mathbf{x}^* A\mathbf{x}$ is 1×1 we find

$$\mathrm{tr}(A\mathbf{x}\mathbf{x}^*) = \mathbf{x}^* A\mathbf{x}.$$

4. Obviously we can calculate A^2 and A^3 by straightforward calculation. From

$$A^2 - (\mathrm{tr}\,A)A + (\det A)I_2 = 0_{2\times 2}$$

we find

$$A^2 = (\mathrm{tr}\,A)A - (\det A)I_2$$

and (multiplying by A)

$$A^3 = (\mathrm{tr}\,A)A^2 - (\det A)A.$$

Since $\mathrm{tr}A = 4$ and $\det A = 3$ we find

$$A^2 = 4A - 3I_2 = \begin{pmatrix} 12 & 8 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 0 & 1 \end{pmatrix},$$

$$A^3 = 4A^2 - 3A = \begin{pmatrix} 36 & 32 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 9 & 6 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 27 & 26 \\ 0 & 1 \end{pmatrix}.$$

A generalization of this technique allows us to express any power of A as a linear combination of A and I_2.