Applied Mathematics 3B

Assignment #2 **7:30, 2 August 2011**

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1. Let

$$A := \hbar\omega \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

Determine the eigenvalues λ_1 and λ_2. Determine the corresponding orthonormal eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Calculate

$$e^{-it\lambda_1/\hbar}\mathbf{x}_1\mathbf{x}_1^* + e^{-it\lambda_2/\hbar}\mathbf{x}_2\mathbf{x}_2^*.$$

2. Solve the differential equation

$$i\hbar\frac{d\boldsymbol{\psi}}{dt} = \left[\hbar\omega \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\right]\boldsymbol{\psi}, \qquad \boldsymbol{\psi}(t), \boldsymbol{\psi}(0) \in \mathbb{C}^2.$$

for $\boldsymbol{\psi}(t)$ in terms of $\boldsymbol{\psi}(0)$.

3. Let A be an arbitrary 2×2 matrix over $\mathbb{C}$ and let $\mathbf{x} \in \mathbb{C}^2$ be arbitrary.

Calculate $\mathbf{x}^* A\mathbf{x}$ and $\operatorname{tr}(A\mathbf{x}\mathbf{x}^*)$ and compare.

Use the *cyclic invariance* of the trace to explain your answer, i.e. $\operatorname{tr}(XYZ) = \operatorname{tr}(ZXY)$ for compatible matrices X, Y and Z (in other words X is $p \times q$, Y is $q \times r$ and Z is $r \times p$ for some $p, q, r \in \mathbb{N}$).

4. Let

$$A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}.$$

Use the Cayley-Hamilton theorem, i.e.

$$A^2 - (\operatorname{tr} A)A + (\det A)I_2 = 0_{2\times 2}$$

where I_2 is the 2×2 identity matrix and $0_{2\times 2}$ is the 2×2 zero matrix, to calculate A^2 and A^3.