Applied Mathematics 3B

Assignment #1 **Solution**

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1. We find

$$\mathbf{x}^T\mathbf{y} = \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 8$$

and

$$\mathbf{x}\mathbf{y}^T = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}.$$

2. The characteristic equation is

$$(\lambda - 1)^2 - 1 = 0$$

i.e. $\lambda_1 = 2$ and $\lambda_2 = 0$. From the eigenvalue equation

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

we find $a = -ib$, i.e. we find a corresponding normalized eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}.$$

From the eigenvalue equation

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0 \cdot \begin{pmatrix} c \\ d \end{pmatrix}$$

we find $c = id$, i.e. we find a corresponding normalized eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}.$$

Consequently

$$2 \cdot \mathbf{x}_1\mathbf{x}_1^* + 0 \cdot \mathbf{x}_2\mathbf{x}_2^* = 2 \cdot \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} = A.$$

This follows from the fact that A is normal, i.e. $AA^* = A^*A$.

3. We seek all 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}$$

with characteristic equation $(\lambda - 1)(\lambda + 1) = \lambda^2 - 1$. The characteristic equation for A is $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$. The trace of A is the sum of the eigenvalues of A so that

$$\mathrm{tr}A = a + d = 0 \qquad \Rightarrow \qquad d = -a$$

and the determinant of A is the product of the eigenvalues of A so that

$$\det A = ad - bc = a(-a) - bc = -1 \qquad \Rightarrow \qquad bc = 1 - a^2.$$

If $bc = 0$ then $a = 1$ or $a = -1$ and we find the matrices

$$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ c & 1 \end{pmatrix}, \qquad b, c \in \mathbb{R}.$$

If $bc \neq 0$ then we find the matrices

$$\begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

4. Examples include

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For enrichment:
We arbitrarily choose the eigenvalue λ with corresponding eigenspace spanned by $(1 \quad 0)^T$. Thus any non-zero vector in the space spanned by $(0 \quad 1)^T$ should not be an eigenvector. Let the 2×2 matrix A be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

By our choice we have

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}.$$

Thus

$$A = \begin{pmatrix} \lambda & b \\ 0 & d \end{pmatrix}, \qquad b, d \in \mathbb{R}.$$

Furthermore

$$A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

should not be an eigenvector, i.e. $b \neq 0$. Thus we find the matrix

$$A = \begin{pmatrix} \lambda & b \\ 0 & d \end{pmatrix}, \qquad b, d \in \mathbb{R},\ b \neq 0.$$

This matrix has *two* eigenvalues, λ and d. Since $b \neq 0$ we find an eigenvector corresponding to the eigenvalue d:

$$\begin{pmatrix} 1 \\ \frac{d-\lambda}{b} \end{pmatrix}.$$

Note that this yields two eigenspaces (which are not orthogonal to each other) unless $d = \lambda$. Thus we find

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \qquad b \in \mathbb{R},\ b \neq 0.$$

Other matrices can be found by starting with a different choice for the eigenspace. Let $\mathbf{x}$ be a normalized eigenvector corresponding to the eigenvalue λ and let $\{\mathbf{x}, \mathbf{x}_\perp\}$ be an orthonormal basis in $\mathbb{R}^2$. Then we find

$$A = \lambda \mathbf{x}\mathbf{x}^T + b\,\mathbf{x}\mathbf{x}_\perp^T + \lambda \mathbf{x}_\perp \mathbf{x}_\perp^T = \lambda I_2 + b\,\mathbf{x}\mathbf{x}_\perp^T, \qquad b \in \mathbb{R}, b \neq 0.$$