Applied Mathematics 3B

Assignment #10 **Solution**

---

1. We have

$$
\begin{aligned}
(U_{QFT,4}\otimes I_4)|\psi_1\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle)\otimes|0\rangle+(U_{QFT,4}|1\rangle)\otimes|1\rangle+(U_{QFT,4}|2\rangle)\otimes|0\rangle+(U_{QFT,4}|3\rangle)\otimes|1\rangle) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right)\otimes|1\rangle\right. \\
&\qquad\left.+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right)\otimes|1\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|1\rangle\right. \\
&\qquad\left.+\left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|1\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1+e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|1\rangle\right) \\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|0\rangle+e^{-i4\pi j/4}|0\rangle+e^{-i2\pi j/4}|1\rangle+e^{-i6\pi j/4}|1\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)+|2\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{0}\to\frac{4}{4}=1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2}=2$ which appears to be the correct underlying periodicity.

$$
\begin{aligned}
(U_{QFT,4}\otimes I_4)|\psi_2\rangle &= \frac{1}{2}(-(U_{QFT,4}|0\rangle)\otimes|0\rangle+(U_{QFT,4}|1\rangle)\otimes|1\rangle-(U_{QFT,4}|2\rangle)\otimes|0\rangle+(U_{QFT,4}|3\rangle)\otimes|1\rangle) \\
&= \frac{1}{4}\left(-\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right)\otimes|1\rangle\right. \\
&\qquad\left.-\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right)\otimes|1\rangle\right) \\
&= \frac{1}{4}\left(-\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|1\rangle\right. \\
&\qquad\left.-\left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|1\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(-1-e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|1\rangle\right)
\end{aligned}
$$

$$= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle \otimes \left(-|0\rangle - e^{-i4\pi j/4}|0\rangle + e^{-i2\pi j/4}|1\rangle + e^{-i6\pi j/4}|1\rangle\right)\right)$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle) + |2\rangle \otimes \frac{1}{\sqrt{2}}(-|0\rangle - |1\rangle)\right).$$

Thus the periodicity is deduced from $\frac{4}{0} \to \frac{4}{4} = 1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2} = 2$ which appears to be the correct underlying periodicity.

Notice that $|\psi_1\rangle$ and $|\psi_2\rangle$ are related by $|0\rangle \to -|0\rangle$.

2.

(a) We have
$$\rho_x = \frac{1}{2}\begin{pmatrix} a & b \\ \bar{b} & 1-a \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1-a & \bar{b} \\ b & a \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & b+\bar{b} \\ b+\bar{b} & 1 \end{pmatrix}.$$

(b) We find
$$\rho_z = \frac{1}{2}\begin{pmatrix} a & b \\ \bar{b} & 1-a \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a & -b \\ -\bar{b} & 1-a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix}.$$

(c) It follows that
$$\rho_{xz} = \frac{1}{2}\begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1-a & 0 \\ 0 & a \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(d) We also find
$$\rho_{zx} = \frac{1}{4}\begin{pmatrix} 1 & b+\bar{b} \\ b+\bar{b} & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -b-\bar{b} \\ -b-\bar{b} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(e) We have
$$\rho_{xz} = \frac{1}{2}(\rho_z + \sigma_x\rho_z\sigma_x^*) = \frac{1}{2}\left(\frac{1}{2}\rho + \frac{1}{2}\sigma_z\rho\sigma_z^* + \frac{1}{2}\sigma_x\rho\sigma_x^* + \frac{1}{2}\sigma_x\sigma_z\rho\sigma_z^*\sigma_x^*\right) = \frac{1}{4}(I_2\rho I_2^* + \sigma_x\rho\sigma_x^* + \sigma_y\rho\sigma_y^* + \sigma_z\rho\sigma_z^*)$$
and
$$\rho_{zx} = \frac{1}{2}(\rho_x + \sigma_z\rho_x\sigma_z^*) = \frac{1}{2}\left(\frac{1}{2}\rho + \frac{1}{2}\sigma_x\rho\sigma_x^* + \frac{1}{2}\sigma_z\rho\sigma_z^* + \frac{1}{2}\sigma_z\sigma_x\rho\sigma_x^*\sigma_z^*\right) = \frac{1}{4}(I_2\rho I_2^* + \sigma_x\rho\sigma_x^* + \sigma_y\rho\sigma_y^* + \sigma_z\rho\sigma_z^*)$$
where we used $\sigma_x\sigma_z = i\sigma_y$ and $\sigma_x\sigma_z\rho\sigma_z^*\sigma_x^* = \sigma_y\rho\sigma_y^*$.

The probability that a measurement that distinguishes between $(1,0)^T$ and $(0,1)^T$ yields the outcome corresponding to $(1,0)^T$ is $1/2$.

Consider the preparation of 4 identical ensembles (mixtures of pure states) each described by the density matrix ρ. We allow each system to evolve where the time evolution (after a fixed time) is given by the unitary matrices I_2, σ_x, σ_y and σ_z. Mixing these ensembles then yields a system described by the density matrix $I_2/2$. Consider the observable
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The expectation value is $1/2$. A measurement (average) of the system yields an average value $x \in [0,1]$, with $x > 1/2$, $x = 1/2$ or $x < 1/2$. Suppose $x > 1/2$, then the state after measurement is now described by the density matrix
$$\begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}.$$

The probability of an outcome describing $(1,0)^T$ is now $x > \frac{1}{2}$. Suppose $x < 1/2$, then the state after measurement is now described by the density matrix
$$\begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}.$$

Applying σ_x to the system yields the density matrix
$$\begin{pmatrix} 1-x & 0 \\ 0 & x \end{pmatrix}.$$

The probability of an outcome describing $(1,0)^T$ is now $1 - x > \frac{1}{2}$.

We can repeat this process, only taking systems where the average outcome is $y > x$ after each step. Thus the probability that the system is in the state $(1,0)^T$ is increasing. This is a very simple purification procedure.