Applied Mathematics 3B

Semester Test #2 **Solution**

1.

(a) We have

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\\ 0 & 1\end{pmatrix}^T \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\\ 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}.$$

The eigenvalues are 1 and $\frac{3}{2}$ with corresponding eigenvectors $(1,0)^T$ and $(0,1)^T$. The columns of U are given by

$$\mathbf{u}_1 = \frac{1}{\sqrt{\frac{3}{2}}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{1}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$$

and the last column chosen to be orthogonal to these two

$$\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\ -1\\ -2\end{pmatrix}.$$

Thus

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\\ 0 & 1\end{pmatrix} = \left[\frac{1}{\sqrt{6}}\begin{pmatrix}\sqrt{2} & \sqrt{3} & 1\\ -\sqrt{2} & \sqrt{3} & -1\\ \sqrt{2} & 0 & -2\end{pmatrix}\right]\begin{pmatrix}\sqrt{\frac{3}{2}} & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^*.$$

(b) The partial trace is given by

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}_A}\begin{pmatrix}\frac{3}{8} & 0 & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & 0 & 0\\ \frac{1}{8} & 0 & \frac{3}{8} & 0\\ 0 & 0 & 0 & \frac{1}{8}\end{pmatrix} &= [(1\ \ 0)\otimes I_2]\begin{pmatrix}\frac{3}{8} & 0 & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & 0 & 0\\ \frac{1}{8} & 0 & \frac{3}{8} & 0\\ 0 & 0 & 0 & \frac{1}{8}\end{pmatrix}\left[\begin{pmatrix}1\\ 0\end{pmatrix}\otimes I_2\right]\\
&+ [(0\ \ 1)\otimes I_2]\begin{pmatrix}\frac{3}{8} & 0 & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & 0 & 0\\ \frac{1}{8} & 0 & \frac{3}{8} & 0\\ 0 & 0 & 0 & \frac{1}{8}\end{pmatrix}\left[\begin{pmatrix}0\\ 1\end{pmatrix}\otimes I_2\right]\\
&= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\frac{3}{8} & 0 & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & 0 & 0\\ \frac{1}{8} & 0 & \frac{3}{8} & 0\\ 0 & 0 & 0 & \frac{1}{8}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix}\\
&+ \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{3}{8} & 0 & \frac{1}{8} & 0\\ 0 & \frac{1}{8} & 0 & 0\\ \frac{1}{8} & 0 & \frac{3}{8} & 0\\ 0 & 0 & 0 & \frac{1}{8}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}\frac{3}{8} & 0\\ 0 & \frac{1}{8}\end{pmatrix} + \begin{pmatrix}\frac{3}{8} & 0\\ 0 & \frac{1}{8}\end{pmatrix} = \begin{pmatrix}\frac{3}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}.
\end{aligned}$$

The eigenvalues are $\frac{3}{4}$ and $\frac{1}{4}$, the von Neumann entropy is

$$-\frac{3}{4}\log_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = 2 - \frac{3}{4}\log_2 3 \approx 0.811278124459.$$

2. Straightforward calculation yields

$$\begin{aligned}
\boldsymbol{\psi} &= U_{CNOT}(U_H \otimes I_2)\left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right] \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} (a_1+a_2)b_1 \\ (a_1+a_2)b_2 \\ (a_1-a_2)b_1 \\ (a_1-a_2)b_2 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} (a_1+a_2)b_1 \\ (a_1+a_2)b_2 \\ (a_1-a_2)b_2 \\ (a_1-a_2)b_1 \end{pmatrix}
\end{aligned}$$

If $\boldsymbol{\psi}$ is separable then there exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ with

$$c_1 d_1 = \frac{2}{\sqrt{2}}(a_1+a_2)b_1, \qquad c_1 d_2 = \frac{2}{\sqrt{2}}(a_1+a_2)b_2, \qquad c_2 d_1 = \frac{2}{\sqrt{2}}(a_1-a_2)b_2, \qquad c_2 d_2 = \frac{2}{\sqrt{2}}(a_1-a_2)b_1.$$

Thus we have

$$c_1 c_2 d_1 d_2 = \frac{1}{2}(a_1^2 - a_2^2)b_1^2 = \frac{1}{2}(a_1^2 - a_2^2)b_2^2$$

which is satisfied if $a_1^2 = a_2^2$ ($a_1 = a_2 = 0$ or $a_1, a_2 \in \{-1, 1\}$), or $b_1^2 = b_2^2$ ($b_1 = b_2 = 0$ or $b_1, b_2 \in \{-1, 1\}$).

Obviously $\boldsymbol{\psi}$ is entangled if $a_1^2 \neq a_2^2$ or $b_1^2 \neq b_2^2$.

3. We have

$$\begin{aligned}
U_{QFT,4}|\psi_1\rangle &= -\frac{1}{4}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|0\rangle + \frac{1}{4}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|1\rangle \\
&\quad -\frac{1}{4}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|2\rangle + \frac{1}{4}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|3\rangle \\
&= -\frac{1}{4}\sum_{j=0}^{3} e^{-i2\pi j\cdot 0/4}|j\rangle + \frac{1}{4}\sum_{j=0}^{3} e^{-i2\pi j\cdot 1/4}|j\rangle - \frac{1}{4}\sum_{j=0}^{3} e^{-i2\pi j\cdot 2/4}|j\rangle + \frac{1}{4}\sum_{j=0}^{3} e^{-i2\pi j\cdot 3/4}|j\rangle \\
&= \frac{1}{4}\sum_{j=0}^{3}\left(-e^{-i2\pi j\cdot 0/4} + e^{-i2\pi j\cdot 1/4} - e^{-i2\pi j\cdot 2/4} + e^{-i2\pi j\cdot 3/4}\right)|j\rangle \\
&= \frac{1}{4}\left[(-1+1-1+1)|0\rangle + (-1-i+1+i)|1\rangle + (-1-1-1-1)|2\rangle + (-1+i+1-i)|3\rangle\right] \\
&= -|2\rangle.
\end{aligned}$$

We identify the period from the non-zero coefficients, namely the coefficient of $|2\rangle$. The period is $4/2 = 2$.

For the second case we find

$$\begin{aligned}
U_{QFT,4}|\psi_2\rangle &= \frac{1}{\sqrt{40}}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|0\rangle + 2\frac{1}{\sqrt{40}}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|1\rangle \\
&+ \frac{1}{\sqrt{40}}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|2\rangle + 2\frac{1}{\sqrt{40}}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/3}|j\rangle\langle k|3\rangle \\
&= \frac{1}{\sqrt{40}}\sum_{j=0}^{3} e^{-i2\pi j\cdot 0/4}|j\rangle + \frac{2}{\sqrt{40}}\sum_{j=0}^{3} e^{-i2\pi j\cdot 1/4}|j\rangle + \frac{1}{\sqrt{40}}\sum_{j=0}^{3} e^{-i2\pi j\cdot 2/4}|j\rangle + \frac{2}{\sqrt{40}}\sum_{j=0}^{3} e^{-i2\pi j\cdot 3/4}|j\rangle \\
&= \frac{1}{\sqrt{40}}\sum_{j=0}^{3}\left(e^{-i2\pi j\cdot 0/4} + 2e^{-i2\pi j\cdot 1/4} + e^{-i2\pi j\cdot 2/4} + 2e^{-i2\pi j\cdot 3/4}\right)|j\rangle
\end{aligned}$$

$$= \frac{1}{\sqrt{40}}[(1+2+1+2)|0\rangle + (1-2i-1+2i)|1\rangle + (1-2+1-2)|2\rangle + (1+2i-1-2i)|3\rangle]$$
$$= \frac{1}{\sqrt{40}}(6|0\rangle - 2|1\rangle).$$

We identify the period from the non-zero coefficients, namely the coefficient of $|0\rangle$ and $|2\rangle$. The period is $4/2 = 2$. we also find the period $4/0 \to 4/4 = 1$ (which is not present in the sequence of coefficients). Notice that

$$|0\rangle + 2|1\rangle + |2\rangle + 2|3\rangle = (1|0\rangle + 11\rangle + 1|2\rangle + 1|3\rangle) + (0|0\rangle + 1|1\rangle + 0|2\rangle + 1|3\rangle).$$

The coefficients of the first summand form a sequence with period 1. The coefficients of the second summand form a sequence with period 2.