UNIVERSITY OF JOHANNESBURG

Applied Mathematics 3B

Semester Test #1 **Solution**

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1. The solution is given by

$$\boldsymbol{\psi}(t) = \exp\left[i\omega t\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right]\boldsymbol{\psi}(0).$$

Let

$$A := i\omega t\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

We have

$$e^A = \sum_{j=0}^{\infty}\frac{A^j}{j!}$$

where

$$A^0 = I_4, \qquad A^1 = A, \qquad A^2 = (i\omega t)^2\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \qquad A^3 = (i\omega t)^3\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Suppose

$$A^{2k} = (i\omega t)^{2k}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

(which is clearly true for $k = 1$). It follows that

$$A^{2(k+1)} = A^{2k+2} = A^{2k}A^2 = (i\omega t)^{2k}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}(i\omega t)^2\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = (i\omega t)^{2(k+1)}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Thus

$$A^{2k} = (i\omega t)^{2k}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

for $k = 1, 2, \ldots$ by induction. Also

$$A^{2k+1} = AA^{2k} = i\omega t\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}(i\omega t)^{2k}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = (i\omega t)^{2k+1}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

for $k = 1, \ldots$. The last expression obviously also holds for $k = 0$. It follows that (splitting the sum over even and odd powers)

$$e^A = \sum_{j=0}^{\infty}\frac{A^j}{j!} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \sum_{k=1}^{\infty}\frac{A^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{A^{2k+1}}{(2k+1)!}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left(\sum_{k=1}^{\infty} \frac{(i\omega t)^{2k}}{(2k)!} \right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \left(\sum_{k=0}^{\infty} \frac{(i\omega t)^{2k}}{(2k+1)!} \right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (\cosh(i\omega t) - 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sinh(i\omega t) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + (\cos(\omega t) - 1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + i\sin(\omega t) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\omega t) & i\sin(\omega t) \\ 0 & 0 & i\sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

Consequently

$$\boldsymbol{\psi}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\omega t) & i\sin(\omega t) \\ 0 & 0 & i\sin(\omega t) & \cos(\omega t) \end{pmatrix} \boldsymbol{\psi}(0).$$

Thus

$$\boldsymbol{\psi}\left(t = \frac{\pi}{2\omega}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \boldsymbol{\psi}(0).$$

For

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

we find

$$\boldsymbol{\psi}\left(t = \frac{\pi}{2\omega}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{\psi}(0).$$

For

$$\boldsymbol{\psi}(t=0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

we find

$$\boldsymbol{\psi}\left(t = \frac{\pi}{2\omega}\right) = \begin{pmatrix} 0 \\ 0 \\ i \\ 0 \end{pmatrix} = i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2. The measurement outcomes are 2 and 1. A set of orthonormal eigenvectors corresponding to these eigenvalues are

$$1: \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad -1: \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Consider the first density operator

$$\rho_1 = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

The probability that measurement yields the measurement outcome 2 is

$$\mathrm{tr}\left(\rho_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]^* \right) = \mathrm{tr}\left(\begin{pmatrix} \cos^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) = \frac{1}{2} \mathrm{tr} \begin{pmatrix} \cos^2\theta & \cos^2\theta \\ \sin^2\theta & \sin^2\theta \end{pmatrix} = \frac{1}{2}.$$

The probability that measurement yields the measurement outcome 1 is

$$\mathrm{tr}\left(\rho_1 \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]^*\right) = \mathrm{tr}\left(\begin{pmatrix}\cos^2\theta & 0\\ 0 & \sin^2\theta\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right) = \frac{1}{2}\mathrm{tr}\begin{pmatrix}\cos^2\theta & -\cos^2\theta\\ -\sin^2\theta & \sin^2\theta\end{pmatrix} = \frac{1}{2}.$$

The expectation value is

$$\mathrm{tr}\left(\rho_1 \frac{1}{2}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\right) = \mathrm{tr}\left(\begin{pmatrix}\cos^2\theta & 0\\ 0 & \sin^2\theta\end{pmatrix}\frac{1}{2}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\right) = \frac{1}{2}\mathrm{tr}\begin{pmatrix}3\cos^2\theta & \cos^2\theta\\ \sin^2\theta & 3\sin^2\theta\end{pmatrix} = \frac{3}{2}.$$

Now consider the second density operator

$$\rho_2 = \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}.$$

The probability that measurement yields the measurement outcome 2 is

$$\begin{aligned}\mathrm{tr}\left(\rho_2 \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right]^*\right) &= \mathrm{tr}\left(\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right)\\ &= \frac{1}{2}\mathrm{tr}\begin{pmatrix}\cos^2\theta+\cos\theta\sin\theta & \cos^2\theta+\cos\theta\sin\theta\\ \cos\theta\sin\theta+\sin^2\theta & \cos\theta\sin\theta+\sin^2\theta\end{pmatrix}\\ &= \frac{1}{2}+\cos\theta\sin\theta = \frac{1}{2}(1+\sin 2\theta).\end{aligned}$$

The probability that measurement yields the measurement outcome 1 is

$$\begin{aligned}\mathrm{tr}\left(\rho_2 \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right]^*\right) &= \mathrm{tr}\left(\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right)\\ &= \frac{1}{2}\mathrm{tr}\begin{pmatrix}\cos^2\theta-\cos\theta\sin\theta & -\cos^2\theta+\cos\theta\sin\theta\\ \cos\theta\sin\theta-\sin^2\theta & -\cos\theta\sin\theta+\sin^2\theta\end{pmatrix}\\ &= \frac{1}{2}-\cos\theta\sin\theta = \frac{1}{2}(1-\sin 2\theta).\end{aligned}$$

The expectation value is

$$\begin{aligned}\mathrm{tr}\left(\rho_2 \frac{1}{2}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\right) &= \mathrm{tr}\left(\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}\frac{1}{2}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\right)\\ &= \frac{1}{2}\mathrm{tr}\begin{pmatrix}3\cos^2\theta+\cos\theta\sin\theta & \cos^2\theta+3\cos\theta\sin\theta\\ \sin^2\theta+3\cos\theta\sin\theta & 3\sin^2\theta+\cos\theta\sin\theta\end{pmatrix}\\ &= \frac{3}{2}+\frac{1}{2}\sin 2\theta.\end{aligned}$$

3. First notice that

$$\begin{aligned}U_H\sigma_z U_H &= U_H|0\rangle\langle 0|U_H - U_H|1\rangle\langle 1|U_H\\ &= \frac{1}{2}(|0\rangle+|1\rangle)(\langle 0|+\langle 1|) - \frac{1}{2}(|0\rangle-|1\rangle)(\langle 0|-\langle 1|)\\ &= \frac{1}{2}(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|+|1\rangle\langle 1|-|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|)\\ &= \sigma_x.\end{aligned}$$

Thus for the right hand size we find

$$\begin{aligned}(U_H\otimes I_2)\Big(I_2\otimes|0\rangle\langle 0|+\sigma_z\otimes|1\rangle\langle 1|\Big)(U_H\otimes I_2) &= (U_H I_2 U_H)\otimes(I_2|0\rangle\langle 0|I_2)+(U_H\sigma_z U_H)\otimes(I_2|1\rangle\langle 1|I_2)\\ &= I_2\otimes|0\rangle\langle 0|+\sigma_x\otimes|1\rangle\langle 1|\end{aligned}$$

where we used that $U_H U_H = I_2$. For the left hand side we find

$$\begin{aligned}(U_H\otimes I_2)\Big(|0\rangle\langle 0|\otimes I_2+|1\rangle\langle 1|\otimes\sigma_z\Big)(U_H\otimes I_2) &= (U_H|0\rangle\langle 0|U_H)\otimes(I_2^3)+(U_H|1\rangle\langle 1|U_H)\otimes(I_2\sigma_z I_2)\\ &= \frac{1}{2}(|0\rangle+|1\rangle)(\langle 0|+\langle 1|)\otimes I_2+\frac{1}{2}(|0\rangle-|1\rangle)(\langle 0|-\langle 1|)\otimes\sigma_z\\ &= \frac{1}{2}(|0\rangle\langle 0|+|1\rangle\langle 1|)\otimes(I_2+\sigma_z)+\frac{1}{2}(|0\rangle\langle 1|+|1\rangle\langle 0|)\otimes(I_2-\sigma_z)\\ &= I_2\otimes|0\rangle\langle 0|+\sigma_x\otimes|1\rangle\langle 1|.\end{aligned}$$

Thus the equality

$$(U_H \otimes I_2)\Big(|0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes \sigma_z\Big)(U_H \otimes I_2) = (I_2 \otimes U_H)\Big(|0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes \sigma_z\Big)(I_2 \otimes U_H)$$

holds.