Applied Mathematics 3B

Exam: November 2009 **Solution**

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1. The measurement outcomes are 1 (multiplicity 3) and -1. A set of orthonormal eigenvectors corresponding to these eigenvalues are

$$1: \quad \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \qquad -1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}.$$

The projection operators are

$$\Pi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}^* + \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}^* + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}^* = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{1}{2}&\frac{1}{2}\\0&0&\frac{1}{2}&\frac{1}{2}\end{pmatrix}$$

and

$$\Pi_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}^* = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&\frac{1}{2}&-\frac{1}{2}\\0&0&-\frac{1}{2}&\frac{1}{2}\end{pmatrix}.$$

For ρ we find

$$\rho = \frac{1}{4}\begin{pmatrix}1&\sin 2\theta&\sin 2\theta&1\\ \sin 2\theta&1&1&\sin 2\theta\\ \sin 2\theta&1&1&\sin 2\theta\\ 1&\sin 2\theta&\sin 2\theta&1\end{pmatrix}.$$

The probability that measurement yields the measurement outcome 1 is

$$\mathrm{tr}(\rho\Pi_1) = \frac{1}{4}\mathrm{tr}\begin{pmatrix}1&\sin 2\theta&\frac{1}{2}+\frac{1}{2}\sin 2\theta&\frac{1}{2}+\frac{1}{2}\sin 2\theta\\ \sin 2\theta&1&\frac{1}{2}+\frac{1}{2}\sin 2\theta&\frac{1}{2}+\frac{1}{2}\sin 2\theta\\ \sin 2\theta&1&\frac{1}{2}+\frac{1}{2}\sin 2\theta&\frac{1}{2}+\frac{1}{2}\sin 2\theta\\ 1&\sin 2\theta&\frac{1}{2}+\frac{1}{2}\sin 2\theta&\frac{1}{2}+\frac{1}{2}\sin 2\theta\end{pmatrix} = \frac{3}{4}+\frac{1}{4}\sin 2\theta.$$

The probability that measurement yields the measurement outcome -1 is

$$\mathrm{tr}(\rho\Pi_{-1}) = \frac{1}{4}\mathrm{tr}\begin{pmatrix}0&0&-\frac{1}{2}+\frac{1}{2}\sin 2\theta&\frac{1}{2}-\frac{1}{2}\sin 2\theta\\ 0&0&\frac{1}{2}-\frac{1}{2}\sin 2\theta&-\frac{1}{2}+\frac{1}{2}\sin 2\theta\\ 0&0&\frac{1}{2}-\frac{1}{2}\sin 2\theta&-\frac{1}{2}+\frac{1}{2}\sin 2\theta\\ 0&0&-\frac{1}{2}+\frac{1}{2}\sin 2\theta&\frac{1}{2}-\frac{1}{2}\sin 2\theta\end{pmatrix} = \frac{1}{4}-\frac{1}{4}\sin 2\theta.$$

The expectation value is

$$\mathrm{tr}\left(\rho_1\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\right) = \frac{1}{4}\mathrm{tr}\begin{pmatrix}1&\sin 2\theta&1&\sin 2\theta\\ \sin 2\theta&1&\sin 2\theta&1\\ \sin 2\theta&1&\sin 2\theta&1\\ 1&\sin 2\theta&1&\sin 2\theta\end{pmatrix} = \frac{1}{2}+\frac{1}{2}\sin 2\theta.$$

2.

(a) The partial trace is given by

$$\mathrm{tr}_{\mathcal{H}_A}\begin{pmatrix}\frac{1}{2}&\frac{1}{4}&0&\frac{1}{4}\\ \frac{1}{4}&\frac{1}{4}&0&0\\ 0&0&0&0\\ \frac{1}{4}&0&0&\frac{1}{4}\end{pmatrix} = [(1\quad 0)\otimes I_2]\begin{pmatrix}\frac{1}{2}&\frac{1}{4}&0&\frac{1}{4}\\ \frac{1}{4}&\frac{1}{4}&0&0\\ 0&0&0&0\\ \frac{1}{4}&0&0&\frac{1}{4}\end{pmatrix}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes I_2\right]$$

$$
\begin{aligned}
&+ \quad [(0 \;\; 1) \otimes I_2] \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes I_2 \right] \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}.
\end{aligned}
$$

The eigenvalues are found from the trace (1) and the determinant ($\frac{3}{16}$), so that

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_1 \lambda_2 = \frac{3}{16}.$$

This yields the quadratic equation

$$\lambda_1^2 - \lambda_1 + \frac{3}{16} = 0,$$

i.e. $\lambda_1 = \frac{3}{4}$ and $\lambda_2 = \frac{1}{4}$. It follows that the von Neumann entropy is

$$-\frac{3}{4} \log_2 \frac{3}{4} - \frac{1}{4} \log_2 \frac{1}{4} = 2 - \frac{3}{4} \log_2 3 \approx 0.811278124459.$$

(b) $\mathbf{x}$ is separable if and only if $x_1 x_4 = x_2 x_3$. We find

$$\widetilde{\mathbf{x}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ -x_3 \\ -x_2 \\ x_1 \end{pmatrix}.$$

Thus

$$\mathbf{x}^T \widetilde{\mathbf{x}} = 2(x_1 x_4 - x_2 x_3)$$

and $x_1 x_4 - x_2 x_3 = 0$ if and only if $\mathbf{x}$ is separable. We have proved that $\mathbf{x}$ is separable if and only if $\mathbf{x}^T \widetilde{\mathbf{x}} = 0$.

Assume $\mathbf{y} := \alpha \mathbf{a} + \mathbf{b}$ is separable. Clearly $\widetilde{\mathbf{y}} = \alpha \widetilde{\mathbf{a}} + \widetilde{\mathbf{b}}$. By assumption

$$\mathbf{y}^T \widetilde{\mathbf{y}} = (\alpha \mathbf{a} + \mathbf{b})^T (\alpha \widetilde{\mathbf{a}} + \widetilde{\mathbf{b}}) = \alpha^2 \mathbf{a}^T \widetilde{\mathbf{a}} + \alpha (\mathbf{a}^T \widetilde{\mathbf{b}} + \mathbf{b}^T \widetilde{\mathbf{a}}) + \mathbf{b}^T \widetilde{\mathbf{b}} = 0.$$

Since $\mathbf{a}$ is entangled on $\mathbb{C}^2 \otimes \mathbb{C}^2$ $\mathbf{a}^T \widetilde{\mathbf{a}} \neq 0$ and we find for the values of α

$$\alpha = \frac{-(\mathbf{a}^T \widetilde{\mathbf{b}} + \mathbf{b}^T \widetilde{\mathbf{a}}) \pm \sqrt{(\mathbf{a}^T \widetilde{\mathbf{b}} + \mathbf{b}^T \widetilde{\mathbf{a}})^2 - 4 \mathbf{a}^T \widetilde{\mathbf{a}} \mathbf{b}^T \widetilde{\mathbf{b}}}}{2 \mathbf{a}^T \widetilde{\mathbf{a}}}.$$

3. We have

$$
\begin{aligned}
(U_{QFT,4} \otimes I_2)|\psi_1\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle) \otimes |1\rangle + (U_{QFT,4}|1\rangle) \otimes |2\rangle + (U_{QFT,4}|2\rangle) \otimes |1\rangle + (U_{QFT,4}|3\rangle) \otimes |2\rangle) \\
&= \frac{1}{4} \left(\left(\sum_{j=0}^{3} \sum_{k=0}^{3} e^{-i2\pi jk/4} |j\rangle\langle k|0\rangle \right) \otimes |1\rangle + \left(\sum_{j=0}^{3} \sum_{k=0}^{3} e^{-i2\pi jk/4} |j\rangle\langle k|1\rangle \right) \otimes |2\rangle \right. \\
&\quad \left. + \left(\sum_{j=0}^{3} \sum_{k=0}^{3} e^{-i2\pi jk/4} |j\rangle\langle k|2\rangle \right) \otimes |1\rangle + \left(\sum_{j=0}^{3} \sum_{k=0}^{3} e^{-i2\pi jk/4} |j\rangle\langle k|3\rangle \right) \otimes |2\rangle \right) \\
&= \frac{1}{4} \left(\left(\sum_{j=0}^{3} |j\rangle \right) \otimes |1\rangle + \left(\sum_{j=0}^{3} e^{-i2\pi j/4} |j\rangle \right) \otimes |2\rangle \right.
\end{aligned}
$$

$$
\begin{aligned}
& \quad + \left(\sum_{j=0}^{3} e^{-i4\pi j/4}|j\rangle\right) \otimes |1\rangle + \left(\sum_{j=0}^{3} e^{-i6\pi j/4}|j\rangle\right) \otimes |2\rangle\Bigg) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1+e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|1\rangle + \left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|2\rangle\right) \\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|1\rangle + e^{-i4\pi j/4}|1\rangle + e^{-i2\pi j/4}|2\rangle + e^{-i6\pi j/4}|2\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|1\rangle+|2\rangle) + |2\rangle\otimes\frac{1}{\sqrt{2}}(|1\rangle-|2\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{0} \to \frac{4}{4} = 1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2} = 2$ which appears to be the correct underlying periodicity.

We have

$$
\begin{aligned}
(U_{QFT,4}\otimes I_2)|\psi_1\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle)\otimes|1\rangle - (U_{QFT,4}|1\rangle)\otimes|0\rangle + (U_{QFT,4}|2\rangle)\otimes|1\rangle - (U_{QFT,4}|3\rangle)\otimes|0\rangle) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right)\otimes|1\rangle - \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right)\otimes|0\rangle\right. \\
&\quad \left. + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right)\otimes|1\rangle - \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right)\otimes|0\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|1\rangle - \left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|0\rangle\right. \\
&\quad \left. + \left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|1\rangle - \left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|0\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1-e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|1\rangle + \left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}-e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|0\rangle\right) \\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|1\rangle - e^{-i4\pi j/4}|1\rangle + e^{-i2\pi j/4}|2\rangle - e^{-i6\pi j/4}|0\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|1\rangle-|0\rangle) + |2\rangle\otimes\frac{1}{\sqrt{2}}(|1\rangle+|0\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{0} \to \frac{4}{4} = 1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2} = 2$ which appears to be the correct underlying periodicity.

4.

(a)

$$U(t)U(t)^* = e^{-iHt}e^{iH^*t} = e^{-iHt}e^{iHt} = e^{-iHt+iHt} = e^{0_n} = I_n.$$

Similarly $U(t)^*U(t)$. The exponents can be summed since $[-iHt, iHt] = 0$.

(b) Differentiating $U(t)$ provides

$$\frac{d}{dt}U(t) = e^{-iHt}(-iH) = -iU(t)H.$$

Multiplying on the left by $iU(t)^*$ yields

$$iU(t)i^*\frac{d}{dt}U(t) = H.$$

(c) We find

$$\frac{d}{dt}U(t) = \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix}.$$

Thus

$$H = i \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$