Applied Mathematics 3B

Assignment #8 **Solution**

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1. We have

$$\begin{aligned}
(U_{QFT,4}\otimes I_2)|\psi_1\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle)\otimes|0\rangle+(U_{QFT,4}|1\rangle)\otimes|1\rangle+(U_{QFT,4}|2\rangle)\otimes|0\rangle+(U_{QFT,4}|3\rangle)\otimes|1\rangle)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right)\otimes|1\rangle\right.\\
&\qquad\left.+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|1\rangle\right.\\
&\qquad\left.+\left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1+e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|0\rangle+e^{-i4\pi j/4}|0\rangle+e^{-i2\pi j/4}|1\rangle+e^{-i6\pi j/4}|1\rangle\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)+|2\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right).
\end{aligned}$$

Thus the periodicity is deduced from $\frac{4}{0}\to\frac{4}{4}=1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2}=2$ which appears to be the correct underlying periodicity.

$$\begin{aligned}
(U_{QFT,4}\otimes I_2)|\psi_2\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle)\otimes|0\rangle+(U_{QFT,4}|1\rangle)\otimes|1\rangle-(U_{QFT,4}|2\rangle)\otimes|0\rangle-(U_{QFT,4}|3\rangle)\otimes|1\rangle)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right)\otimes|1\rangle\right.\\
&\qquad\left.-\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right)\otimes|0\rangle-\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|1\rangle\right.\\
&\qquad\left.-\left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|0\rangle-\left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1-e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}-e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|1\rangle\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|0\rangle - e^{-i4\pi j/4}|0\rangle + e^{-i2\pi j/4}|1\rangle - e^{-i6\pi j/4}|1\rangle\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(|1\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) + |3\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{1}=4$ and $\frac{4}{3}$ which does not yield an integer period. We can write $|\psi_2\rangle$ in two different ways

$$
\begin{aligned}
|\psi_2\rangle &= \frac{1}{2}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle + (-|2\rangle)\otimes|0\rangle + (-|3\rangle)\otimes|1\rangle)\\
|\psi_2\rangle &= \frac{1}{2}(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle + |2\rangle\otimes(-|0\rangle) + |3\rangle\otimes(-|1\rangle)
\end{aligned}
$$

In the first case we have a periodic sequence, but our quantum Fourier transform must be adapted to $|2\rangle\to-|2\rangle$ and $|3\rangle\to-|3\rangle$ (i.e. to find the period we must apply the correct quantum Fourier transform). In the second case we note that we have four different states in the sequence leading to the conclusion that the sequence has period 4 (i.e. as we found from applying the standard quantum Fourier transform).

2. We find

$$
|\psi\rangle\langle\psi| = (|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle)(\langle 0|\otimes\langle 1| - \langle 1|\otimes\langle 0|) = |0\rangle\langle 0|\otimes|1\rangle\langle 1| - |0\rangle\langle 1|\otimes|1\rangle\langle 0| - |1\rangle\langle 0|\otimes|0\rangle\langle 1| + |1\rangle\langle 1|\otimes|0\rangle\langle 0|.
$$

Thus

$$
\begin{aligned}
\sum_{j=0}^{1}(\langle j|\otimes I_2)\,|\psi\rangle\langle\psi|\,(|j\rangle\otimes I_2) &= (\langle 0|\otimes I_2)(|0\rangle\langle 0|\otimes|1\rangle\langle 1| - |0\rangle\langle 1|\otimes|1\rangle\langle 0| - |1\rangle\langle 0|\otimes|0\rangle\langle 1| + |1\rangle\langle 1|\otimes|0\rangle\langle 0|)(|0\rangle|\otimes I_2)\\
&+ (\langle 1|\otimes I_2)(|0\rangle\langle 0|\otimes|1\rangle\langle 1| - |0\rangle\langle 1|\otimes|1\rangle\langle 0| - |1\rangle\langle 0|\otimes|0\rangle\langle 1| + |1\rangle\langle 1|\otimes|0\rangle\langle 0|)(|1\rangle|\otimes I_2)\\
&= (\langle 0|0\rangle\langle 0|0\rangle)\otimes|1\rangle\langle 1| - (\langle 0|0\rangle\langle 1|0\rangle)\otimes|1\rangle\langle 0|\\
&- (\langle 0|1\rangle\langle 0|0\rangle)\otimes|0\rangle\langle 1| + (\langle 0|1\rangle\langle 1|0\rangle)\otimes|0\rangle\langle 0|\\
&+ (\langle 1|0\rangle\langle 0|1\rangle)\otimes|1\rangle\langle 1| - (\langle 1|0\rangle\langle 1|1\rangle)\otimes|1\rangle\langle 0|\\
&- (\langle 1|1\rangle\langle 0|1\rangle)\otimes|0\rangle\langle 1| + (\langle 1|1\rangle\langle 1|1\rangle)\otimes|0\rangle\langle 0|\\
&= |1\rangle\langle 1| + |0\rangle\langle 0| = I_2
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=0}^{1}(I_2\otimes\langle j|)\,|\psi\rangle\langle\psi|\,(I_2\otimes|j\rangle) &= (I_2\otimes\langle 0|)(|0\rangle\langle 0|\otimes|1\rangle\langle 1| - |0\rangle\langle 1|\otimes|1\rangle\langle 0| - |1\rangle\langle 0|\otimes|0\rangle\langle 1| + |1\rangle\langle 1|\otimes|0\rangle\langle 0|)(I_2\otimes|0\rangle|)\\
&+ (I_2\otimes\langle 1|)(|0\rangle\langle 0|\otimes|1\rangle\langle 1| - |0\rangle\langle 1|\otimes|1\rangle\langle 0| - |1\rangle\langle 0|\otimes|0\rangle\langle 1| + |1\rangle\langle 1|\otimes|0\rangle\langle 0|)(I_2\otimes|1\rangle|)\\
&= |0\rangle\langle 0|\otimes(\langle 0|1\rangle\langle 1|0\rangle) - |0\rangle\langle 1|\otimes(\langle 0|1\rangle\langle 0|0\rangle)\\
&- |1\rangle\langle 0|\otimes(\langle 0|0\rangle\langle 1|0\rangle) + |1\rangle\langle 1|\otimes(\langle 0|0\rangle\langle 0|0\rangle)\\
&+ |0\rangle\langle 0|\otimes(\langle 1|1\rangle\langle 1|1\rangle) - |0\rangle\langle 1|\otimes(\langle 1|1\rangle\langle 0|1\rangle)\\
&- |1\rangle\langle 0|\otimes(\langle 1|0\rangle\langle 1|1\rangle) + |1\rangle\langle 1|\otimes(\langle 1|0\rangle\langle 0|1\rangle)\\
&= |1\rangle\langle 1| + |0\rangle\langle 0| = I_2
\end{aligned}
$$