Applied Mathematics 3B

Assignment #6 **Solution**

1.

(a) We show that $|0'\rangle$ and $|1'\rangle$ are normalized and orthogonal:

$$\langle 0'|0'\rangle = \Big(\overline{a}\langle 0| + \overline{b}\langle 1|\Big)\Big(a|0\rangle + b|1\rangle\Big) = |a|^2\langle 0|0\rangle + \overline{a}b\langle 0|1\rangle + \overline{b}a\langle 1|0\rangle + |b|^2\langle 1|1\rangle = |a|^2 + |b|^2 = 1$$

$$\langle 1'|1'\rangle = \Big(a\langle 1| - b\langle 0|\Big)\Big(\overline{a}|1\rangle - \overline{b}|0\rangle\Big) = |a|^2\langle 1|1\rangle - b\overline{a}\langle 0|1\rangle - a\overline{b}\langle 1|0\rangle + |b|^2\langle 0|0\rangle = |a|^2 + |b|^2 = 1$$

$$\langle 0'|1'\rangle = \Big(\overline{a}\langle 0| + \overline{b}\langle 1|\Big)\Big(\overline{a}|1\rangle - \overline{b}|0\rangle\Big) = \overline{a}^2\langle 0|1\rangle + \overline{b}\overline{a}\langle 1|1\rangle - \overline{a}\overline{b}\langle 0|0\rangle - \overline{b}^2\langle 1|0\rangle = \overline{b}\overline{a} - \overline{a}\overline{b} = 0.$$

(b) We find

$$\begin{aligned}
|0'\rangle\langle 0'| &= |a|^2|0\rangle\langle 0| + a\overline{b}|0\rangle\langle 1| + \overline{a}b|1\rangle\langle 0| + |b|^2|1\rangle\langle 1| \\
|0'\rangle\langle 1'| &= -ab|0\rangle\langle 0| + a^2|0\rangle\langle 1| - b^2|1\rangle\langle 0| + ab|1\rangle\langle 1| \\
|1'\rangle\langle 0'| &= -\overline{a}\overline{b}|0\rangle\langle 0| - \overline{b}^2|0\rangle\langle 1| + \overline{a}^2|1\rangle\langle 0| + \overline{a}\overline{b}|1\rangle\langle 1| \\
|1'\rangle\langle 1'| &= |b|^2|0\rangle\langle 0| - a\overline{b}|0\rangle\langle 1| - \overline{a}b|1\rangle\langle 0| + |a|^2|1\rangle\langle 1|
\end{aligned}$$

Consequently

$$\begin{aligned}
|0'\rangle\langle 0'| \otimes |0'\rangle\langle 0'| &= |0\rangle\langle 0| \otimes \big(|a|^4|0\rangle\langle 0| + |a^2|a\overline{b}|0\rangle\langle 1| + |a^2|\overline{a}b|1\rangle\langle 0| + |ab|^2|1\rangle\langle 1|\big) \\
&+ |0\rangle\langle 1| \otimes \big(|a|^2a\overline{b}|0\rangle\langle 0| + (a\overline{b})^2|0\rangle\langle 1| + |a\overline{b}|^2|1\rangle\langle 0| + |b|^2a\overline{b}|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 0| \otimes \big(|a|^2\overline{a}b|0\rangle\langle 0| + |a\overline{b}|^2|0\rangle\langle 1| + (\overline{a}b)^2|1\rangle\langle 0| + \overline{a}b|b|^2|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 1| \otimes \big(|ab|^2|0\rangle\langle 0| + a\overline{b}|b|^2|0\rangle\langle 1| + \overline{a}b|b|^2|1\rangle\langle 0| + |b|^4|1\rangle\langle 1|\big) \\
|1'\rangle\langle 1'| \otimes |1'\rangle\langle 1'| &= |0\rangle\langle 0| \otimes \big(|b|^4|0\rangle\langle 0| - |b^2|a\overline{b}|0\rangle\langle 1| - |b^2|\overline{a}b|1\rangle\langle 0| + |ab|^2|1\rangle\langle 1|\big) \\
&+ |0\rangle\langle 1| \otimes \big(-|b|^2a\overline{b}|0\rangle\langle 0| + (a\overline{b})^2|0\rangle\langle 1| + |a\overline{b}|^2|1\rangle\langle 0| - |a|^2a\overline{b}|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 0| \otimes \big(-|b|^2\overline{a}b|0\rangle\langle 0| + |a\overline{b}|^2|0\rangle\langle 1| + (\overline{a}b)^2|1\rangle\langle 0| - \overline{a}b|a|^2|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 1| \otimes \big(|ab|^2|0\rangle\langle 0| - a\overline{b}|a|^2|0\rangle\langle 1| - \overline{a}b|a|^2|1\rangle\langle 0| + |a|^4|1\rangle\langle 1|\big) \\
|0'\rangle\langle 1'| \otimes |1'\rangle\langle 0'| &= |0\rangle\langle 0| \otimes \big(|ab|^2|0\rangle\langle 0| + |b|^2a\overline{b}|0\rangle\langle 1| - |a|^2\overline{a}b|1\rangle\langle 0| - |ab|^2|1\rangle\langle 1|\big) \\
&+ |0\rangle\langle 1| \otimes \big(-|a|^2a\overline{b}|0\rangle\langle 0| - (a\overline{b})^2|0\rangle\langle 1| + |a|^4|1\rangle\langle 0| + |a|^2a\overline{b}|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 0| \otimes \big(|a|^2a\overline{b}|0\rangle\langle 0| + |b|^4|0\rangle\langle 1| - (\overline{a}b)^2|1\rangle\langle 0| - |b|^2\overline{a}b|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 1| \otimes \big(-|ab|^2|0\rangle\langle 0| - |b|^2a\overline{b}|0\rangle\langle 1| + |a|^2\overline{a}b|1\rangle\langle 0| + |ab|^2|1\rangle\langle 1|\big) \\
|1'\rangle\langle 0'| \otimes |0'\rangle\langle 1'| &= |0\rangle\langle 0| \otimes \big(|ab|^2|0\rangle\langle 0| - |a|^2a\overline{b}|0\rangle\langle 1| + |b|^2\overline{a}b|1\rangle\langle 0| - |ab|^2|1\rangle\langle 1|\big) \\
&+ |0\rangle\langle 1| \otimes \big(|a|^2\overline{a}b|0\rangle\langle 0| - (a\overline{b})^2|0\rangle\langle 1| + |b|^4|1\rangle\langle 0| - |b|^2a\overline{b}|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 0| \otimes \big(-|a|^2\overline{a}b|0\rangle\langle 0| + |a|^4|0\rangle\langle 1| - (\overline{a}b)^2|1\rangle\langle 0| + |a|^2\overline{a}b|1\rangle\langle 1|\big) \\
&+ |1\rangle\langle 1| \otimes \big(-|ab|^2|0\rangle\langle 0| + |a|^2a\overline{b}|0\rangle\langle 1| - |b|^2\overline{a}b|1\rangle\langle 0| + |ab|^2|1\rangle\langle 1|\big)
\end{aligned}$$

The sum of these expressions is

$$\begin{aligned}
P' &= \Big(|a|^4 + 2|ab|^2 + |b|^4\Big)|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \Big(|a|^4 + 2|a\overline{b}|^2 + |b|^4\Big)|0\rangle\langle 1| \otimes |1\rangle\langle 0| \\
&+ \Big(|a|^4 + 2|a\overline{b}|^2 + |b|^4\Big)|1\rangle\langle 0| \otimes |0\rangle\langle 1| + \Big(|a|^4 + 2|ab|^2 + |b|^4\Big)|1\rangle\langle 1| \otimes |1\rangle\langle 1| \\
&= \Big(|a|^2 + |b|^2\Big)^2|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \Big(|a|^2 + |\overline{b}|^2\Big)^2|0\rangle\langle 1| \otimes |1\rangle\langle 0| \\
&+ \Big(|a|^2 + |\overline{b}|^2\Big)^2|1\rangle\langle 0| \otimes |0\rangle\langle 1| + \Big(|a|^2 + |b|^2\Big)^2|1\rangle\langle 1| \otimes |1\rangle\langle 1| \\
&= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| = P.
\end{aligned}$$

The last steps follows by noting that $|\overline{b}| = |b|$ and $|a|^2 + |b|^2 = 1$. The operator P is the swap gate.

2.

(a) We have

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2.$$

The eigenvalue is 2 (so that the singular value is $\sqrt{2}$) with corresponding orthonormal eigenvector 1. Thus we set $V = 1$

$$\Sigma = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix},$$

and the first column U_1 of U is given by

$$U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot V = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The second column U_2 is chosen to be orthogonal to the first (for example by Gram-Schmidt orthogonalization)

$$U_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Consequently

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} (1)^T.$$

(b) We have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues are 0 and 1 (so that the singular values are also 0 and 1) with corresponding orthonormal eigenvectors

$$\sigma_2 = 0 : \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \sigma_1 = 1 : \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus we set $V = 1$

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and the first column U_1 of U is given by

$$U_1 = \frac{1}{1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since the second singular value is zero the second column U_2 is chosen to be orthogonal to the first (for example by Gram-Schmidt orthogonalization)

$$U_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Consequently

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T.$$

(c) From

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$$

we find

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

with eigenvalues 0 and 2 (singular values 0 and $\sqrt{2}$) and corresponding eigenvectors

$$\sigma_2 = 0 : \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \sigma_1 = \sqrt{2} : \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The first column U_1 of U is given by

$$U_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

Since the second singular value is zero the remaining columns of U are chosen to be orthonormal to each other (for example by Gram-Schmidt orthonormalization)

$$U_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad U_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad U_4 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Other choices are possible. Thus we find

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T$$

which can also be written as

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left(\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \left(\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \left((1) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)^T.$$

Other choices for U lead to different singular value decompositions. We find that a singular value decomposition of a Kronecker product is given by the Kronecker products of the V, Σ and U of the matrices appearing in the Kronecker product.