Applied Mathematics 3B

Assignment #5 **Solution**

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1. We have

$$
\begin{aligned}
I_2|0\rangle &= |0\rangle\langle 0|0\rangle + |1\rangle\langle 1|0\rangle = |0\rangle \\
I_2|1\rangle &= |0\rangle\langle 0|1\rangle + |1\rangle\langle 1|1\rangle = |1\rangle \\
\langle 0|I_2 &= \langle 0|0\rangle\langle 0| + \langle 0|1\rangle\langle 1| = \langle 0| \\
\langle 1|I_2 &= \langle 1|0\rangle\langle 0| + \langle 1|1\rangle\langle 1| = \langle 1|
\end{aligned}
$$

and for U_{NOT}

$$
\begin{aligned}
U_{NOT}|0\rangle &= |1\rangle\langle 0|0\rangle + |0\rangle\langle 1|0\rangle = |1\rangle \\
U_{NOT}|1\rangle &= |1\rangle\langle 0|1\rangle + |0\rangle\langle 1|1\rangle = |0\rangle \\
\langle 0|U_{NOT} &= \langle 0|1\rangle\langle 0| + \langle 0|0\rangle\langle 1| = \langle 1| \\
\langle 1|U_{NOT} &= \langle 1|1\rangle\langle 0| + \langle 1|0\rangle\langle 1| = \langle 0|
\end{aligned}
$$

It follows that

$$
\begin{aligned}
U_{CNOT21}U_{CNOT12} &= (I_2 \otimes |0\rangle\langle 0| + U_{NOT} \otimes |1\rangle\langle 1|)(|0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{NOT}) \\
&= (I_2 \otimes |0\rangle\langle 0|)(|0\rangle\langle 0| \otimes I_2) + (I_2 \otimes |0\rangle\langle 0|)(|1\rangle\langle 1| \otimes U_{NOT}) \\
&+ (U_{NOT} \otimes |1\rangle\langle 1|)(|0\rangle\langle 0| \otimes I_2) + (U_{NOT} \otimes |1\rangle\langle 1|)(|1\rangle\langle 1| \otimes U_{NOT}) \\
&= (I_2|0\rangle\langle 0|) \otimes (|0\rangle\langle 0|I_2) + (I_2|1\rangle\langle 1|) \otimes (|0\rangle\langle 0|U_{NOT}) \\
&+ (U_{NOT}|0\rangle\langle 0|) \otimes (|1\rangle\langle 1|I_2) + (U_{NOT}|1\rangle\langle 1|) \otimes (|1\rangle\langle 1|U_{NOT}) \\
&= ((I_2|0\rangle)\langle 0|) \otimes (|0\rangle(\langle 0|I_2)) + ((I_2|1\rangle)\langle 1|) \otimes (|0\rangle(\langle 0|U_{NOT})) \\
&+ ((U_{NOT}|0\rangle)\langle 0|) \otimes (|1\rangle(\langle 1|I_2)) + ((U_{NOT}|1\rangle)\langle 1|) \otimes (|1\rangle(\langle 1|U_{NOT})) \\
&= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0|
\end{aligned}
$$

Finally

$$
\begin{aligned}
U_{CNOT12}U_{CNOT21}U_{CNOT12} &= (|0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{NOT}) \\
&\times (|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0|) \\
&= (|0\rangle\langle 0|0\rangle\langle 0|) \otimes (I_2|0\rangle\langle 0|) + (|0\rangle\langle 0|1\rangle\langle 1|) \otimes (I_2|0\rangle\langle 1|) \\
&+ (|0\rangle\langle 0|1\rangle\langle 0|) \otimes (I_2|1\rangle\langle 1|) + (|0\rangle\langle 0|0\rangle\langle 1|) \otimes (I_2|1\rangle\langle 0|) \\
&+ (|1\rangle\langle 1|0\rangle\langle 0|) \otimes (U_{NOT}|0\rangle\langle 0|) + (|1\rangle\langle 1|1\rangle\langle 1|) \otimes (U_{NOT}|0\rangle\langle 1|) \\
&+ (|1\rangle\langle 1|1\rangle\langle 0|) \otimes (U_{NOT}|1\rangle\langle 1|) + (|1\rangle\langle 1|0\rangle\langle 1|) \otimes (U_{NOT}|1\rangle\langle 0|) \\
&= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| = U_{SWAP}.
\end{aligned}
$$

2. For $\alpha = 1$ we find

$$
\begin{aligned}
U_D(\alpha = 1) &= \Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|\Big) \otimes I_2 + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \left[i\cos\left(\frac{\pi}{2}\right) I_2 + \sin\left(\frac{\pi}{2}\right)\sigma_x\right] \\
&= \Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|\Big) \otimes I_2 + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \sigma_x
\end{aligned}
$$

where $\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1| = U_{NOT}$.

(a) It follows that

$$\begin{aligned}
U_D(\alpha=1)(|a\rangle\otimes|1\rangle\otimes|1\rangle) &= (|0\rangle\langle 0|a\rangle)\otimes(|0\rangle\langle 0|1\rangle)\otimes(I_2|1\rangle)+(|0\rangle\langle 0|a\rangle)\otimes(|1\rangle\langle 1|1\rangle)\otimes(I_2|1\rangle)\\
&+ (|1\rangle\langle 1|a\rangle)\otimes(|0\rangle\langle 0|1\rangle)\otimes(I_2|1\rangle)+(|1\rangle\langle 1|a\rangle)\otimes(|1\rangle\langle 1|1\rangle)\otimes(U_{NOT}|1\rangle)\\
&= \delta_{a,0}|0\rangle\otimes|1\rangle\otimes|1\rangle+\delta_{a,1}|1\rangle\otimes|1\rangle\otimes|0\rangle\\
&= \begin{cases}|0\rangle\otimes|1\rangle\otimes|1\rangle & a=0\\ |1\rangle\otimes|1\rangle\otimes|0\rangle & a=1\end{cases}\\
&= |a\rangle\otimes|1\rangle\otimes|\overline{a}\rangle.
\end{aligned}$$

(b) The next calculation proceeds similarly.

$$\begin{aligned}
U_D(\alpha=1)(|a\rangle\otimes|b\rangle\otimes|0\rangle) &= (|0\rangle\langle 0|a\rangle)\otimes(|0\rangle\langle 0|b\rangle)\otimes(I_2|0\rangle)+(|0\rangle\langle 0|a\rangle)\otimes(|1\rangle\langle 1|b\rangle)\otimes(I_2|0\rangle)\\
&+ (|1\rangle\langle 1|a\rangle)\otimes(|0\rangle\langle 0|b\rangle)\otimes(I_2|0\rangle)+(|1\rangle\langle 1|a\rangle)\otimes(|1\rangle\langle 1|b\rangle)\otimes(U_{NOT}|0\rangle)\\
&= \delta_{a,0}\delta_{b,0}|0\rangle\otimes|0\rangle\otimes|0\rangle+\delta_{a,0}\delta_{b,1}|0\rangle\otimes|1\rangle\otimes|0\rangle\\
&+ \delta_{a,1}\delta_{b,0}|1\rangle\otimes|0\rangle\otimes|0\rangle+\delta_{a,1}\delta_{b,1}|1\rangle\otimes|1\rangle\otimes|1\rangle\\
&= \begin{cases}|0\rangle\otimes|0\rangle\otimes|0\rangle & a=0,b=0\\ |0\rangle\otimes|1\rangle\otimes|0\rangle & a=0,b=1\\ |1\rangle\otimes|0\rangle\otimes|0\rangle & a=1,b=0\\ |1\rangle\otimes|1\rangle\otimes|1\rangle & a=1,b=1\end{cases}\\
&= |a\rangle\otimes|b\rangle\otimes|a\cdot b\rangle.
\end{aligned}$$

(c) Consider

$$\begin{aligned}
&(I_2\otimes I_2\otimes U_D(\alpha=1))(U_D(\alpha=1)\otimes I_2\otimes I_2)(|a\rangle\otimes|b\rangle\otimes|0\rangle\otimes|1\rangle\otimes|1\rangle)\\
&\quad= (I_2\otimes I_2\otimes U_D(\alpha=1))\,[U_D(\alpha=1)(|a\rangle\otimes|b\rangle\otimes|0\rangle)\otimes(I_2|1\rangle)\otimes(I_2|1\rangle)]\\
&\quad= (I_2\otimes I_2\otimes U_D(\alpha=1))(|a\rangle\otimes|b\rangle\otimes|a\cdot b\rangle\otimes|1\rangle\otimes|1\rangle)\\
&\quad= (I_2|a\rangle)\otimes(I_2|b\rangle)\otimes[U_D(\alpha=1)(|a\cdot b\rangle\otimes|1\rangle\otimes|1\rangle)]\\
&\quad= |a\rangle\otimes|b\rangle\otimes|a\cdot b\rangle\otimes|1\rangle\otimes|\overline{a\cdot b}\rangle.
\end{aligned}$$

Thus

$$(I_2\otimes I_2\otimes U_D(\alpha=1))(U_D(\alpha=1)\otimes I_2\otimes I_2)$$

applied to $|a\rangle\otimes|b\rangle\otimes|0\rangle\otimes|1\rangle\otimes|1\rangle$ yields the desired operation in the last qubit.