UNIVERSITEIT VAN JOHANNESBURG

Toegepaste Wiskunde 3B

Taak #5 **7:30, 18 Augustus 2009**

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1. Beskou die Hilbertruimte $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. Laat

$$|0\rangle, |1\rangle \in \mathbb{C}^2, \qquad \langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0.$$

Bereken

$$U_{CNOT12} U_{CNOT21} U_{CNOT12}$$

waar

$$I_2 := |0\rangle\langle 0| + |1\rangle\langle 1|, \qquad U_{NOT} := |1\rangle\langle 0| + |0\rangle\langle 1|$$
$$U_{CNOT12} := |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{NOT}, \qquad U_{CNOT21} := I_2 \otimes |0\rangle\langle 0| + U_{NOT} \otimes |1\rangle\langle 1|$$
$$U_{SWAP} := |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|.$$

2. Laat

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Deutsch se hek

$$U_D(\alpha) := \Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|\Big) \otimes I_2 + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \left[i \cos\left(\alpha \frac{\pi}{2}\right) I_2 + \sin\left(\alpha \frac{\pi}{2}\right) \sigma_x \right]$$

word toegepas op elemente van die Hilbertruimte $\mathbb{C}^8$. In matriks vorm, Deutsch se hek word deur

$$U_D(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\left(\alpha\frac{\pi}{2}\right) & i\cos\left(\alpha\frac{\pi}{2}\right) \end{pmatrix}$$

gegee.

(a) Wys dat

$$U_D(\alpha = 1)(|a\rangle \otimes |1\rangle \otimes |1\rangle) = |a\rangle \otimes |1\rangle \otimes |\overline{a}\rangle, \quad a \in \{0, 1\}$$

waar $\overline{a}$ deur die volgende tabel gegee word.

a	$\overline{a}$
0	1
1	0

(b) Wys dat

$$U_D(\alpha = 1)(|a\rangle \otimes |b\rangle \otimes |0\rangle) = |a\rangle \otimes |b\rangle \otimes |a \cdot b\rangle, \quad a, b \in \{0, 1\}$$

waar $a \cdot b$ deur die volgende tabel gegee word.

a	b	$a \cdot b$
0	0	0
0	1	0
1	0	0
1	1	1

(c) Hoe kan $\overline{a \cdot b}$ geimplimenteer word? **Wenk:** Beskou die Hilbertruimte $\mathbb{C}^{32}$, d.w.s. 5 qubits.

UNIVERSITY OF JOHANNESBURG

Applied Mathematics 3B

Assignment #5 **7:30, 18 August 2009**

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1. Consider the Hilbert space $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$. Let

$$|0\rangle, |1\rangle \in \mathbb{C}^2, \qquad \langle 0|0\rangle = \langle 1|1\rangle = 1, \qquad \langle 0|1\rangle = \langle 1|0\rangle = 0.$$

Calculate

$$U_{CNOT12} U_{CNOT21} U_{CNOT12}$$

where

$$I_2 := |0\rangle\langle 0| + |1\rangle\langle 1|, \qquad U_{NOT} := |1\rangle\langle 0| + |0\rangle\langle 1|$$
$$U_{CNOT12} := |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes U_{NOT}, \qquad U_{CNOT21} := I_2 \otimes |0\rangle\langle 0| + U_{NOT} \otimes |1\rangle\langle 1|$$
$$U_{SWAP} := |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|.$$

2. Let

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Deutsch's gate

$$U_D(\alpha) := \Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|\Big) \otimes I_2 + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \left[i\cos\left(\alpha\frac{\pi}{2}\right) I_2 + \sin\left(\alpha\frac{\pi}{2}\right)\sigma_x\right]$$

acts on elements of the Hilbert space $\mathbb{C}^8$. In matrix form, Deutsch's gate is given by

$$U_D(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\left(\alpha\frac{\pi}{2}\right) & i\cos\left(\alpha\frac{\pi}{2}\right) \end{pmatrix}$$

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(a) Show that

$$U_D(\alpha = 1)(|a\rangle \otimes |1\rangle \otimes |1\rangle) = |a\rangle \otimes |1\rangle \otimes |\overline{a}\rangle, \quad a \in \{0, 1\}$$

where $\overline{a}$ is given by the table

a	$\overline{a}$
0	1
1	0

(b) Show that

$$U_D(\alpha = 1)(|a\rangle \otimes |b\rangle \otimes |0\rangle) = |a\rangle \otimes |b\rangle \otimes |a \cdot b\rangle, \quad a, b \in \{0, 1\}$$

where $a \cdot b$ is given by the table

a	b	$a \cdot b$
0	0	0
0	1	0
1	0	0
1	1	1

(c) How can $\overline{a \cdot b}$ be implemented? **Hint:** Consider the Hilbert space $\mathbb{C}^{32}$, i.e. 5 qubits.
