Applied Mathematics 3B

Assignment #4 **Solution**

---

1. We find

$$(\sigma_x \otimes \sigma_x)^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the matrix describes an observable. The measurement outcomes are the eigenvalues of the matrix. The characteristic equation is

$$|\lambda I_4 - \sigma_x \otimes \sigma_x| = \begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} + \begin{vmatrix} 0 & \lambda & -1 \\ 0 & -1 & \lambda \\ -1 & 0 & 0 \end{vmatrix} = \lambda(\lambda^3 - \lambda) + (-\lambda^2 + 1) = (\lambda^2 - 1)^2.$$

Thus the eigenvalues are ± 1 each with multiplicity 2. The eigenstates corresponding to the eigenvalue 1 are the non-zero vectors $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$ where $(\sigma_x \otimes \sigma_x)\mathbf{x} = \mathbf{x}$, i.e. $x_1 = x_4$ and $x_2 = x_3$. To find out whether these states are separable, we suppose that they are:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \\ x_1 \end{pmatrix}.$$

It follows that $ac = x_1$, $ad = x_2$, $bc = x_2$ and $bd = x_1$. Multiplying pairs of equations we find $abcd = x_1^2 = x_2^2$. Thus when $x_1^2 \neq x_2^2$ the eigenstates are not separable. If $x_1^2 = x_2^2$ then $x_1 \neq 0$ and $x_2 \neq 0$. It follows that $a, b, c, d \neq 0$ so that $a = \frac{x_1}{c} = \frac{x_2}{d}$, $b = \frac{x_2}{c} = \frac{x_1}{d}$, and $c = \frac{x_1}{x_2} d$. Thus when $x_1^2 = x_2^2$ the eigenstates are separable.

The eigenstates corresponding to the eigenvalue -1 are the non-zero vectors $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$ where $(\sigma_x \otimes \sigma_x)\mathbf{x} = -\mathbf{x}$, i.e. $x_1 = -x_4$ and $x_2 = -x_3$. To find out whether these states are separable, we suppose that they are:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{pmatrix}.$$

It follows that $ac = x_1$, $ad = x_2$, $bc = -x_2$ and $bd = -x_1$. Multiplying pairs of equations we find $abcd = -x_1^2 = -x_2^2$. Thus when $x_1^2 \neq x_2^2$ the eigenstates are not separable. If $x_1^2 = x_2^2$ then $x_1 \neq 0$ and $x_2 \neq 0$. It follows that $a, b, c, d \neq 0$ so that $a = \frac{x_1}{c} = \frac{x_2}{d}$, $b = -\frac{x_2}{c} = -\frac{x_1}{d}$, and $c = \frac{x_1}{x_2} d$. Thus when $x_1^2 = x_2^2$ the eigenstates are separable.

We choose orthonormal eigenvectors corresponding to the eigenvalues

$$1: \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad -1: \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

Other choices are possible and do not change the result of the calculation. The probability of measurement outcome 1 is

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|^2 + \left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|^2 = \frac{1}{2}.$$

The probability of measurement outcome -1 is

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|^2 + \left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|^2 = \frac{1}{2}.$$

2. The measurement outcomes are 1 and -1. A set of orthonormal eigenvectors corresponding to these eigenvalues are

$$1: \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad -1: \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Consider the first state: $e^{i\theta}/\sqrt{2}\,(1 \quad 1)^T$. The density operator is

$$\rho_1 = \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]^* = \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{e^{-i\theta}}{\sqrt{2}} (1 \quad 1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Notice that ρ_1 is independent of θ. The probability that measurement yields the measurement outcome 1 is

$$\mathrm{tr} \left(\rho_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]^* \right) = \mathrm{tr} \left(\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) = \frac{1}{4} \mathrm{tr} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 1.$$

The probability that measurement yields the measurement outcome -1 is

$$\mathrm{tr} \left(\rho_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]^* \right) = \mathrm{tr} \left(\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) = \frac{1}{4} \mathrm{tr} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

The expectation value is

$$\mathrm{tr} \left(\rho_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \mathrm{tr} \left(\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \frac{1}{2} \mathrm{tr} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 1.$$

This is consistent with the measurement outcomes $1 \cdot 1 + 0 \cdot (-1) = 1$.

Now consider the state: $1/\sqrt{2}\,(1 \quad e^{i\theta})^T$. The density operator is

$$\rho_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \right]^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \frac{1}{\sqrt{2}} (1 \quad e^{-i\theta}) = \frac{1}{2} \begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix}.$$

The probability that measurement yields the measurement outcome 1 is

$$\begin{aligned} \mathrm{tr} \left(\rho_2 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]^* \right) &= \mathrm{tr} \left(\frac{1}{2} \begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \\ &= \frac{1}{4} \mathrm{tr} \begin{pmatrix} 1 + e^{-i\theta} & 1 + e^{-i\theta} \\ 1 + e^{i\theta} & 1 + e^{i\theta} \end{pmatrix} \\ &= \frac{1}{2} \left(1 + \frac{1}{2} \left(e^{i\theta} + e^{-i\theta} \right) \right) = \frac{1}{2} (1 + \cos\theta). \end{aligned}$$

The probability that measurement yields the measurement outcome -1 is

$$\begin{aligned} \mathrm{tr} \left(\rho_2 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]^* \right) &= \mathrm{tr} \left(\frac{1}{2} \begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) \\ &= \frac{1}{4} \mathrm{tr} \begin{pmatrix} 1 - e^{-i\theta} & -1 + e^{-i\theta} \\ -1 + e^{i\theta} & 1 - e^{i\theta} \end{pmatrix} \\ &= \frac{1}{2} \left(1 - \frac{1}{2} \left(e^{i\theta} + e^{-i\theta} \right) \right) = \frac{1}{2} (1 - \cos\theta). \end{aligned}$$

The expectation value is

$$\mathrm{tr} \left(\rho_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \mathrm{tr} \left(\frac{1}{2} \begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \frac{1}{2} \mathrm{tr} \begin{pmatrix} e^{-i\theta} & 1 \\ 1 & e^{i\theta} \end{pmatrix} = \cos\theta.$$

This is consistent with the measurement outcomes $\frac{1}{2}(1+\cos\theta)\cdot 1 + \frac{1}{2}(1-\cos\theta)\cdot(-1) = \cos\theta$.

3. Let

$$\mathbf{a} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

We have

$$\mathbf{a}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{a} = \begin{pmatrix} a_1b_1 \\ a_1b_2 \\ a_2b_1 \\ a_2b_2 \end{pmatrix} - \begin{pmatrix} b_1a_1 \\ b_1a_2 \\ b_2a_1 \\ b_2a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_1b_2 - a_2b_1 \\ -(a_1b_2 - a_2b_1) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \det\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} \\ -\det\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} \\ 0 \end{pmatrix}.$$

Setting $\mathbf{a}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{a} = \mathbf{0}$ yields $\det\begin{pmatrix}\mathbf{a} & \mathbf{b}\end{pmatrix} = 0$, i.e. $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. Consequently $\mathbf{a} = c\mathbf{b}$ for some $c \in \mathbb{C}$ and $\mathbf{b} \in \mathbb{C}^2$.