Applied Mathematics 3B

Assignment #3 **Solution**

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1. Obviously

$$\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -1 \end{pmatrix}^* = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & -1 \end{pmatrix}.$$

Thus the matrix describes an observable. The measurement outcomes are the eigenvalues of the matrix. The characteristic equation is $(\lambda - 1)(\lambda^2 - 1) - (\lambda - 1) = 0$. Thus the eigenvalues are 1, $\sqrt{2}$ and $-\sqrt{2}$. (Check the sum of the eigenvalues against the trace of the matrix: 1, and the product of the eigenvalues against the determinant of the matrix: -2.) A set of orthonormal eigenvectors corresponding to the eigenvalues is

$$1: \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \sqrt{2}: \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1 \\ 0 \\ (1-\sqrt{2})i \end{pmatrix}, \qquad -\sqrt{2}: \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1 \\ 0 \\ (1+\sqrt{2})i \end{pmatrix}.$$

The probability that measurement yields the measurement outcome 1 is

$$\left\| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^* \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 = \frac{1}{3}.$$

After the measurement yields 1 the system is in the state

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^* \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{normalize}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The probability that measurement yields the measurement outcome $\sqrt{2}$ is

$$\begin{aligned} \left\| \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ 0 \\ (1-\sqrt{2})i \end{pmatrix}^* \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 &= \left\| \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 & 0 & -(1-\sqrt{2})i \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 \\ &= \frac{1}{3} \left| \frac{1}{\sqrt{4-2\sqrt{2}}} (1 - (1-\sqrt{2})i) \right|^2 = \frac{1}{3}. \end{aligned}$$

After the measurement yields $\sqrt{2}$ the system is in the state

$$\begin{aligned} &\frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ 0 \\ (1-\sqrt{2})i \end{pmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ 0 \\ (1-\sqrt{2})i \end{pmatrix}^* \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{4-2\sqrt{2}} \begin{pmatrix} 1 & 0 & -(1-\sqrt{2})i \\ 0 & 0 & 0 \\ (1-\sqrt{2})i & 0 & 3-2\sqrt{2} \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \frac{1}{4-2\sqrt{2}} \frac{1}{\sqrt{3}} \begin{pmatrix} 1-(1-\sqrt{2})i \\ 0 \\ 3-2\sqrt{2}+(1-\sqrt{2})i \end{pmatrix} \xrightarrow{\text{normalize}} \frac{1-(1-\sqrt{2})i}{4-2\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ (1-\sqrt{2})i \end{pmatrix}. \end{aligned}$$

The probability that measurement yields the measurement outcome $-\sqrt{2}$ is

$$\begin{aligned} \left\| \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ 0 \\ (1+\sqrt{2})i \end{pmatrix}^* \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 &= \left\| \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 & 0 & -(1+\sqrt{2})i \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 \\ &= \frac{1}{3} \left| \frac{1}{\sqrt{4+2\sqrt{2}}} (1 - (1+\sqrt{2})i) \right|^2 = \frac{1}{3}. \end{aligned}$$

After the measurement yields $-\sqrt{2}$ the system is in the state

$$\begin{aligned}
&\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix}1\\0\\(1+\sqrt{2})i\end{pmatrix}\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix}1\\0\\(1+\sqrt{2})i\end{pmatrix}^*\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\\
&= \frac{1}{4+2\sqrt{2}}\begin{pmatrix}1 & 0 & -(1+\sqrt{2})i\\ 0 & 0 & 0\\ (1+\sqrt{2})i & 0 & 3+2\sqrt{2}\end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\\
&= \frac{1}{4+2\sqrt{2}}\frac{1}{\sqrt{3}}\begin{pmatrix}1-(1+\sqrt{2})i\\0\\3+2\sqrt{2}+(1+\sqrt{2})i\end{pmatrix} \overset{\text{normalize}}{\longrightarrow} \frac{1-(1+\sqrt{2})i}{4+2\sqrt{2}}\begin{pmatrix}1\\0\\(1+\sqrt{2})i\end{pmatrix}.
\end{aligned}$$

2. The measurement outcomes are 1 and -1. A set of orthonormal eigenvectors corresponding to these eigenvalues are

$$1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad -1: \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Consider the first state: $e^{i\theta}/\sqrt{2}\,(1 \quad 1)^T$. The probability that measurement yields the measurement outcome 1 is

$$\left\|\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}^*\frac{e^{i\theta}}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right\|^2 = \frac{1}{4}|2|^2 = 1$$

independent of θ. Thus the state after measurement is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}^*\frac{e^{i\theta}}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{e^{i\theta}}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

Obviously the probability that measurement yields the measurement outcome -1 is 0. The factor $e^{i\theta}$ plays no roll in the measurement. It is a global phase factor.

Now consider the state: $1/\sqrt{2}\,(1 \quad e^{i\theta})^T$. The probability that measurement yields the measurement outcome 1 is

$$\left\|\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}^*\frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\theta}\end{pmatrix}\right\|^2 = \frac{1}{4}|1+e^{i\theta}|^2 = \frac{1}{4}|1+\cos\theta+i\sin\theta|^2 = \frac{1}{4}((1+\cos\theta)^2+\sin^2\theta) = \frac{1}{2}(1+\cos\theta)$$

which depends on θ. Here $e^{i\theta}$ is a local phase factor. Thus the state after measurement yields the outcome 1 is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}^*\frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\theta}\end{pmatrix} = \frac{1+e^{i\theta}}{\sqrt{8}}\begin{pmatrix}1\\1\end{pmatrix}$$

before normalization. Note that $\theta=\pi$ yields the zero vector, so we cannot normalize this state in general.

The probability that measurement yields the measurement outcome -1 is

$$\left\|\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}^*\frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\theta}\end{pmatrix}\right\|^2 = \frac{1}{4}|1-e^{i\theta}|^2 = \frac{1}{4}|1-\cos\theta-i\sin\theta|^2 = \frac{1}{4}((1-\cos\theta)^2+\sin^2\theta) = \frac{1}{2}(1-\cos\theta)$$

which depends on θ. Thus the state after measurement yields the outcome -1 is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}^*\frac{1}{\sqrt{2}}\begin{pmatrix}1\\e^{i\theta}\end{pmatrix} = \frac{1-e^{i\theta}}{\sqrt{8}}\begin{pmatrix}1\\-1\end{pmatrix}$$

before normalization. Note that $\theta=0$ yields the zero vector, so we cannot normalize this state in general.

3. Since A and B commute $[A,B]=AB-BA=0$. Let $\mathbf{x}$ be an eigenvector of A corresponding to the eigenvalue λ. Then

$$A(B\mathbf{x}) = B(A\mathbf{x}) = \lambda(B\mathbf{x}).$$

Thus $B\mathbf{x}$ is either $\mathbf{0}$ or $B\mathbf{x}$ is also an eigenvector of A corresponding to λ.

Since the eigenspace of A corresponding to λ is one dimensional,

$$B\mathbf{x} = \mu\mathbf{x}$$

for some $\mu \in \mathbb{R}$. Thus $\mathbf{x}$ is an eigenvector of B corresponding to the eigenvalue μ.

It follows that A and B share one-dimensional eigenspaces and that measurement of A and B are compatible in the sense that performing the measurement described by B after the measurement described by A does not change the system after the measurement described by A (and vice-versa). If the system is in state described by an eigenstate of A (and consequently also B), then the results of measurement are independent of the order of measurement (A then B or B then A).