Applied Mathematics 3B

Assignment #2 **Solution**

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1. We have

$$\begin{array}{ccc} \mathbf{x}_1^*\mathbf{x}_1 = 1 & \mathbf{x}_1^*\mathbf{x}_2 = 0 & \mathbf{x}_1^*\mathbf{x}_3 = 0 \\ \mathbf{x}_2^*\mathbf{x}_1 = 0 & \mathbf{x}_2^*\mathbf{x}_2 = 1 & \mathbf{x}_2^*\mathbf{x}_3 = 0 \\ \mathbf{x}_3^*\mathbf{x}_1 = 0 & \mathbf{x}_3^*\mathbf{x}_2 = 0 & \mathbf{x}_3^*\mathbf{x}_3 = 1 \end{array}$$

i.e. we have an orthonormal basis. We find

$$\mathbf{x}_1\mathbf{x}_1^* + \mathbf{x}_2\mathbf{x}_2^* + \mathbf{x}_3\mathbf{x}_3^* = \begin{pmatrix} \frac{1}{4} & \sqrt{\frac{3}{32}} & \sqrt{\frac{3}{32}} \\ \sqrt{\frac{3}{32}} & \frac{3}{8} & \frac{3}{8} \\ \sqrt{\frac{3}{32}} & \frac{3}{8} & \frac{3}{8} \end{pmatrix} + \begin{pmatrix} \frac{3}{4} & -\sqrt{\frac{3}{32}} & -\sqrt{\frac{3}{32}} \\ -\sqrt{\frac{3}{32}} & \frac{1}{8} & \frac{1}{8} \\ -\sqrt{\frac{3}{32}} & \frac{1}{8} & \frac{1}{8} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = I_3.$$

2. We have

$$\begin{array}{ccc} \mathbf{x}_1^*\mathbf{x}_1 = 1 & \mathbf{x}_1^*\mathbf{x}_2 = 0 & \mathbf{x}_1^*\mathbf{x}_3 = 0 \\ \mathbf{x}_2^*\mathbf{x}_1 = 0 & \mathbf{x}_2^*\mathbf{x}_2 = 1 & \mathbf{x}_2^*\mathbf{x}_3 = 0 \\ \mathbf{x}_3^*\mathbf{x}_1 = 0 & \mathbf{x}_3^*\mathbf{x}_2 = 0 & \mathbf{x}_3^*\mathbf{x}_3 = 1 \end{array}$$

i.e. we have an orthonormal basis. We find

$$\mathbf{x}_1\mathbf{x}_1^* + \mathbf{x}_2\mathbf{x}_2^* + \mathbf{x}_3\mathbf{x}_3^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos^2\theta & 0 & \cos\theta\sin\theta \\ 0 & 0 & 0 \\ \cos\theta\sin\theta & 0 & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & 0 & -\cos\theta\sin\theta \\ 0 & 0 & 0 \\ -\cos\theta\sin\theta & 0 & \cos^2\theta \end{pmatrix} = I_3.$$

3. We have

$$\lambda_1 = \hbar\omega, \quad \mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \lambda_2 = -\hbar\omega, \quad \mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Thus we find

$$\begin{aligned} e^{-it\lambda_1/\hbar}\mathbf{x}_1\mathbf{x}_1^* + e^{-it\lambda_2/\hbar}\mathbf{x}_2\mathbf{x}_2^* &= \frac{e^{-it\omega}}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} + \frac{e^{it\omega}}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{e^{-it\omega}+e^{it\omega}}{2} & i\dfrac{-e^{-it\omega}+e^{it\omega}}{2} \\ i\dfrac{e^{-it\omega}-e^{it\omega}}{2} & \dfrac{e^{-it\omega}+e^{it\omega}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}. \end{aligned}$$

4. The solution is given by

$$\boldsymbol{\psi}(t) = e^{-i\left[\hbar\omega\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]t/\hbar}\boldsymbol{\psi}(0) = e^{-i\omega t\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}\boldsymbol{\psi}(0).$$

Let

$$A := -i\omega t\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We have

$$e^A = \sum_{j=0}^{\infty}\frac{A^j}{j!}$$

where

$$A^0 = I_2, \qquad A^1 = A, \qquad A^2 = (-i\omega t)^2 I_2.$$

Suppose $A^{2k} = (-i\omega t)^{2k} I_2$ (which is clearly true for $k = 0$ and $k = 1$). It follows that

$$A^{2(k+1)} = A^{2k+2} = A^{2k}A^2 = (-i\omega t)^{2k} I_2 (-i\omega t)^2 I_2 = (-i\omega t)^{2(k+1)} I_2.$$

Thus $A^{2k} = (-i\omega t)^{2k} I_2$ for $k = 0, 1, \ldots$ by induction. Also $A^{2k+1} = AA^{2k} = (-i\omega t)^{2k} A$ for $k = 0, 1, \ldots$. It follows that (splitting the sum over even and odd powers)

$$\begin{aligned}
e^A &= \sum_{j=0}^{\infty} \frac{A^j}{j!} = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \left(\frac{(-i\omega t)^{2k}}{(2k)!} I_2 \right) + \sum_{k=0}^{\infty} \left(\frac{(-i\omega t)^{2k}}{(2k+1)!} A \right) = \left(\sum_{k=0}^{\infty} \frac{(-i\omega t)^{2k}}{(2k)!} \right) I_2 + \left(\sum_{k=0}^{\infty} \frac{(-i\omega t)^{2k}}{(2k+1)!} \right) A \\
&= \left(\sum_{k=0}^{\infty} \frac{(-i\omega t)^{2k}}{(2k)!} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\sum_{k=0}^{\infty} \frac{(-i\omega t)^{2k+1}}{(2k+1)!} \right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \left(\sum_{k=0}^{\infty} \frac{(-1)^k \omega t)^{2k}}{(2k)!} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i \left(\sum_{k=0}^{\infty} \frac{(-1)^k (\omega t)^{2k+1}}{(2k+1)!} \right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \cos \omega t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin \omega t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}.
\end{aligned}$$

where we used $(-i)^{2k} = (-1)^k$. Consequently

$$\boldsymbol{\psi}(t) = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \boldsymbol{\psi}(0).$$