Applied Mathematics 3B

Assignment #1 **Solution**

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1. Examples include

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For enrichment:
If we consider Hermitian matrices with the eigenvalues 1 and -1, we find the matrices

$$\mathbf{x}_1\mathbf{x}_1^T - \mathbf{x}_2\mathbf{x}_2^T$$

where $\{\mathbf{x}_1, \mathbf{x}_2\}$ is an orthonormal basis in $\mathbb{R}^2$. For example, choosing

$$\left\{\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \quad \left\{\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad \left\{\mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

yields the three examples given above. Choosing an arbitrary orthonormal basis

$$\left\{\mathbf{x}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \mathbf{x}_2 = \pm\begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}, \ \theta \in \mathbb{R}\right\}$$

yields the matrix

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

In general, any real valued 2×2 matrix A with $\mathrm{tr}A = 0$ and $\det A = -1$, i.e.

$$A = \begin{pmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{pmatrix} \qquad a, b \in \mathbb{R},\ b \neq 0 \qquad \text{or} \qquad A = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

has 1 and -1 as eigenvalues.

2. Examples include

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For enrichment:
We arbitrarily choose the eigenvalue λ with corresponding eigenspace spanned by $(1 \quad 0)^T$. Thus any non-zero vector in the space spanned by $(0 \quad 1)^T$ should not be an eigenvector. Let the 2×2 matrix A be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

By our choice we have

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}.$$

Thus

$$A = \begin{pmatrix} \lambda & b \\ 0 & d \end{pmatrix}, \qquad b, d \in \mathbb{R}.$$

Furthermore

$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

should not be an eigenvector, i.e. $b \neq 0$. Thus we find the matrix

$$A = \begin{pmatrix} \lambda & b \\ 0 & d \end{pmatrix}, \qquad b, d \in \mathbb{R},\ b \neq 0.$$

This matrix has *two* eigenvalues, λ and d. Since $b \neq 0$ we find an eigenvector corresponding to the eigenvalue d:

$$\begin{pmatrix} 1 \\ \frac{d-\lambda}{b} \end{pmatrix}.$$

Note that this yields two eigenspaces (which are not orthogonal to each other) unless $d = \lambda$. Thus we find

$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \qquad b \in \mathbb{R},\ b \neq 0.$$

Other matrices can be found by starting with a different choice for the eigenspace. Let $\mathbf{x}$ be a normalized eigenvector corresponding to the eigenvalue λ and let $\{\mathbf{x}, \mathbf{x}_\perp\}$ be an orthonormal basis in $\mathbb{R}^2$. Then we find

$$A = \lambda \mathbf{x}\mathbf{x}^T + b\,\mathbf{x}\mathbf{x}_\perp^T + \lambda \mathbf{x}_\perp \mathbf{x}_\perp^T = \lambda I_2 + b\,\mathbf{x}\mathbf{x}_\perp^T, \qquad b \in \mathbb{R}, b \neq 0.$$

3. We seek all 2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}$$

with characteristic equation $(\lambda - 0)(\lambda - 1) = \lambda^2 - \lambda$. The characteristic equation for A is $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$. The trace of A is the sum of the eigenvalues of A so that

$$\mathrm{tr}A = a + d = 1 \qquad \Rightarrow \qquad d = 1 - a$$

and the determinant of A is the product of the eigenvalues of A so that

$$\det A = ad - bc = a(1-a) - bc = 0 \qquad \Rightarrow \qquad bc = a(1-a).$$

If $bc = 0$ then $a = 0$ or $a = 1$ and we find the matrices

$$\begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}, \qquad b, c \in \mathbb{R}.$$

If $bc \neq 0$ then we find the matrices

$$\begin{pmatrix} a & b \\ \frac{a(1-a)}{b} & 1-a \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

4. The characteristic equation is

$$\lambda^2 - 1 = 0$$

i.e. $\lambda_1 = 1$ and $\lambda_2 = -1$. From the eigenvalue equation

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

we find $a = -ib$, i.e. we find a corresponding normalized eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}.$$

From the eigenvalue equation

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = -1 \cdot \begin{pmatrix} c \\ d \end{pmatrix}$$

we find $c = id$, i.e. we find a corresponding normalized eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}.$$

Consequently

$$1 \cdot \mathbf{x}_1 \mathbf{x}_1^* + (-1) \cdot \mathbf{x}_2 \mathbf{x}_2^* = \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & \frac{i}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = A.$$

This follows from the fact that A is normal, i.e. $AA^* = A^*A$.
