Applied Mathematics 3B

Assignment #10 **Solution**

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1. The critical points are found for $f'(x) = 0$ where

$$f(x) = -x\log_2 x - (1-x)\log_2(1-x), \qquad x \in [0,1].$$

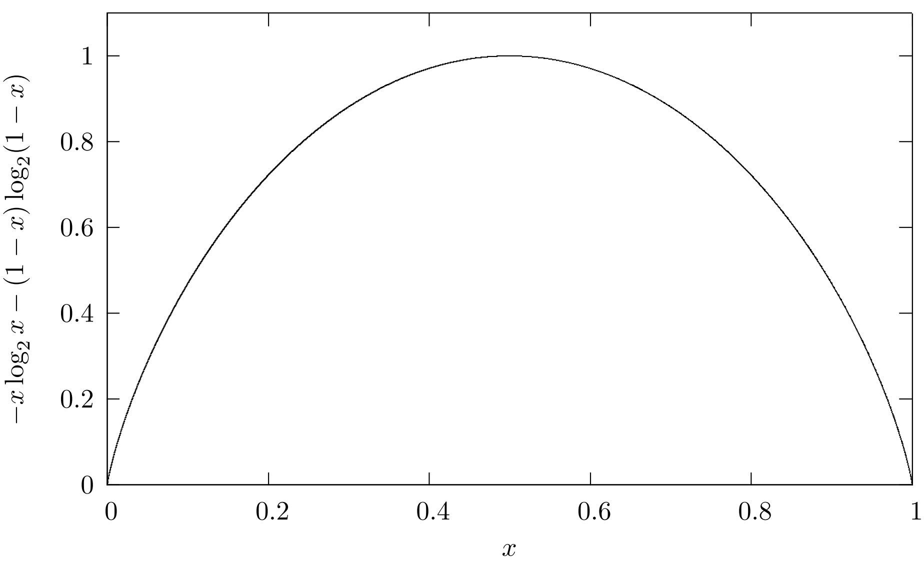
We find the equation

$$\frac{df}{dx} = -\log_2 x - \frac{1}{\ln 2} + \log_2(1-x) + \frac{1}{\ln 2} = \log_2 \frac{1-x}{x} = 0 \quad \Rightarrow \quad \frac{1-x}{x} = 1.$$

Thus $x = \frac{1}{2}$. Since

$$\frac{d^2 f}{dx^2} = -\frac{1}{x\ln 2} - \frac{1}{(1-x)\ln 2} < 0 \qquad \forall x \in [0,1]$$

we found a (global) maximum of $f(\frac{1}{2}) = 1$. The boundaries give global minima at $x = 0$ and $x = 1$ of $f(0) = f(1) = 0$. A graph of the function is given below.



2. Amongst others, two examples are

$$\mathbf{x} = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}.$$

The first is separable, while the second is not. Now we solve the problem in general. Let $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$. Thus we determine whether

$$\begin{pmatrix}\frac{1}{\sqrt{2}} + x_1\\ x_2\\ x_3\\ \frac{1}{\sqrt{2}} + x_4\end{pmatrix}$$

is entangled. The vector is separable if and only if

$$\left(\frac{1}{\sqrt{2}} + x_1\right)\left(\frac{1}{\sqrt{2}} + x_4\right) = x_2 x_3 \qquad \Rightarrow \qquad \frac{1}{2} + \frac{1}{\sqrt{2}}(x_1 + x_4) = (x_2 x_3 - x_1 x_4).$$

We consider two cases:

1. $x_1 = -\frac{1}{\sqrt{2}}$: then $x_2 = 0$ or $x_3 = 0$ and we find the solutions

$$\left\{ \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ t \\ 0 \\ s \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ t \\ s \end{pmatrix}, \qquad t, s \in \mathbf{C} \right\}.$$

These solutions are separable if and only if $s = 0$.

2. $x_1 \neq -\frac{1}{\sqrt{2}}$: then

$$x_4 = \frac{x_2 x_3}{x_1 + \frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{2}}$$

and we find the solutions

$$\begin{pmatrix} t \\ u \\ v \\ \frac{uv}{t+\frac{1}{\sqrt{2}}} - \frac{1}{\sqrt{2}} \end{pmatrix}, \quad t, u, v \in \mathbf{C}$$

These solutions are separable if and only if

$$uv = \frac{tuv}{t + \frac{1}{\sqrt{2}}} - \frac{t}{\sqrt{2}} \qquad \Rightarrow \qquad uv = -t\left(t + \frac{1}{\sqrt{2}}\right).$$

Setting $t = u = v = 0$ yields the separable example chosen above, and setting $t = \frac{1}{\sqrt{2}}$ and $u = v = 0$ yields the entangled example chosen above.

3. Straightforward calculation yields

$$\begin{aligned} F(\rho, |\psi\rangle) &= \sqrt{\langle\psi|\rho|\psi\rangle} \\ &= \sqrt{\langle\psi| \left(p_0 |\psi\rangle\langle\psi| + \sum_{j=1}^{m} p_j \, X_j |\psi\rangle\langle\psi| X_j^* \right) |\psi\rangle} \\ &= \sqrt{p_0 \langle\psi|\psi\rangle\langle\psi|\psi\rangle + \sum_{j=1}^{m} p_j \, \langle\psi|X_j|\psi\rangle\langle\psi|X_j^*|\psi\rangle} \\ &= \sqrt{p_0 + \sum_{j=1}^{m} p_j \, \langle\psi|X_j|\psi\rangle(\langle\psi|X_j|\psi\rangle)^*} \\ &= \sqrt{p_0 + \sum_{j=1}^{m} p_j |\langle\psi|X_j|\psi\rangle|^2} \end{aligned}$$

where we used $\langle\psi|\psi\rangle = 1$ and $(\langle\psi|X_j|\psi\rangle)^* = \overline{\langle\psi|X_j|\psi\rangle}$.

For $m = 1$, $n = 2$, and $|\psi\rangle = |0\rangle$ and $X_1 = \sigma_x = U_{NOT}$ we have $p_1 = 1 - p_0$ and

$$F(\rho, |\psi\rangle = \sqrt{p_0 + (1 - p_0)|\langle 0|\sigma_x|0\rangle|^2} = \sqrt{p_0 + (1 - p_0)|\langle 0|1\rangle|^2} = \sqrt{p_0}.$$

The minimum fidelity for $p_0 \in [0, 1]$ is obviously 0.

For $m = 1$, $n = 2$, $p_0 = p_1 = \frac{1}{2}$,

$$|\psi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle, \qquad \theta \in [0, 2\pi]$$

and $X_1 = \sigma_x = U_{NOT}$ we have

$$\begin{aligned} F(\rho, |\psi\rangle &= \sqrt{\frac{1}{2} + \frac{1}{2}|(\cos\theta\langle 0| + \sin\theta\langle 1|)\sigma_x(\cos\theta|0\rangle + \sin\theta|1\rangle|^2} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}|(\cos\theta\langle 0| + \sin\theta\langle 1|)(\cos\theta|1\rangle + \sin\theta|0\rangle)|^2} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}\sin^2 2\theta.} \end{aligned}$$

The minimum $\frac{1}{\sqrt{2}}$ is obviously achieved when $\sin 2\theta = 0$, i.e. for

$$\theta \in \left\{ 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2},\ 2\pi \right\}.$$

For $m = 1$, $n = 8$, $p_0 = p_1 = \frac{1}{2}$,

$$|\psi\rangle = \cos\theta\,|0\rangle \otimes |0\rangle \otimes |0\rangle + \sin\theta\,|1\rangle \otimes |1\rangle \otimes |1\rangle, \qquad \theta \in [0, 2\pi]$$

and $X_1 = I_2 \otimes I_2 \otimes \sigma_x$ we have

$$\begin{aligned} F(\rho, |\psi\rangle &= \sqrt{\frac{1}{2} + \frac{1}{2}|(\cos\theta\langle 0| \otimes \langle 0| \otimes \langle 0| + \sin\theta\langle 1| \otimes \langle 1| \otimes \langle 1|)X_1(\cos\theta|0\rangle \otimes |0\rangle \otimes |0\rangle + \sin\theta|1\rangle \otimes |1\rangle \otimes |1\rangle|^2} \\ &= \sqrt{\frac{1}{2} + \frac{1}{2}|(\cos\theta\langle 0| \otimes \langle 0| \otimes \langle 0| + \sin\theta\langle 1| \otimes \langle 1| \otimes \langle 1|)(\cos\theta|0\rangle \otimes |0\rangle \otimes |1\rangle + \sin\theta|1\rangle \otimes |1\rangle \otimes |0\rangle|^2} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

independent of θ.