University of Johannesburg

Applied Mathematics 3B

Assignment #8 **Solutions**

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1. One example is

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Another example is

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the next solutions we will find a general form for U.

2. If both $|\phi\rangle$ and $|\psi\rangle$ are the zero vector the U is an arbitrary 2×2 unitary matrix. If $|\phi\rangle$ is the zero vector and $|\psi\rangle$ is non-zero then U does not exist. Similarly $|\psi\rangle$ is the zero vector and $|\phi\rangle$ is non-zero then U does not exist. This can be generalized by noting that

$$\| \, |\psi\rangle \, \|^2 = \langle\psi|\psi\rangle = (\langle\phi|U^*)(U|\phi\rangle) = \langle\phi|(U^*U)|\phi\rangle = \langle\phi|\phi\rangle = \| \, |\phi\rangle \, \|^2$$

i.e. $|\psi\rangle$ and $|\phi\rangle$ must have the same norm to ensure the existence of U with the desired property. From the solution above U need not be unique. In the following we assume $\| \, |\psi\rangle \, \| = \| \, |\phi\rangle \, \| \neq 0$.

(a) Let

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad |\phi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.$$

Note that $2(a)$ can be solved easily from $2(b)$ using

$$|\phi_\perp\rangle = \frac{1}{\||\phi\rangle\|} \begin{pmatrix} -\overline{\phi_2} \\ \overline{\phi_1} \end{pmatrix}, \qquad |\psi_\perp\rangle = \frac{1}{\||\psi\rangle\|} \begin{pmatrix} -\overline{\psi_2} \\ \overline{\psi_1} \end{pmatrix}.$$

Here we derive the same result without using 2(b).

Since U is unitary we have $UU^* = U^*U = I_2$, i.e.

$$|u_1|^2 + |u_2|^2 = |u_3|^2 + |u_4|^2 = |u_1|^2 + |u_3|^2 = |u_2|^2 + |u_4|^2 = 1 \quad \Rightarrow \quad |u_1| = |u_4|, \ |u_2| = |u_3|$$

$$u_1\overline{u_3} + u_2\overline{u_4} = u_1\overline{u_2} + u_3\overline{u_4} = 0 \tag{1}$$

and from $U|\phi\rangle = |\psi\rangle$ we find

$$u_1\phi_1 + u_2\phi_2 = \psi_1, \qquad u_3\phi_1 + u_4\phi_2 = \psi_2. \tag{2}$$

Let $u_1 = |u_1|e^{i\alpha}$, $u_2 = |u_2|e^{i\beta}$, $u_3 = |u_2|e^{i\theta}$ and $u_4 = |u_1|e^{i\gamma}$ be the polar forms of the variables. Consequently $u_4 = e^{i(\gamma-\alpha)}u_1$ and $u_3 = e^{i(\theta-\beta)}u_2$. The equation (1) takes the form

$$|u_1||u_2| \left(e^{i(\alpha-\beta)} + e^{i(\theta-\gamma)}\right) = |u_1||u_2| \left(e^{i(\alpha-\theta)} + e^{i(\beta-\gamma)}\right) = 0. \tag{3}$$

Inserting the polar form into the first equation of (2) and multiplying by $\overline{\phi_1}$ yields

$$|u_1|\phi_1 = e^{-i\alpha}\psi_1 - |u_2|e^{i(\beta-\alpha)}\phi_2 \quad \Rightarrow \quad |u_1||\phi_1|^2 = e^{-i\alpha}\psi_1\overline{\phi_1} - |u_2|e^{i(\beta-\alpha)}\phi_2\overline{\phi_1}.$$

Inserting the polar form into the second equation of (2) and multiplying by $\overline{\phi_2}$ yields

$$|u_1|\phi_2 = e^{-i\gamma}\psi_2 - |u_2|e^{i(\theta-\gamma)}\phi_1 \quad \Rightarrow \quad |u_1||\phi_2|^2 = e^{-i\gamma}\psi_2\overline{\phi_2} - |u_2|e^{i(\theta-\gamma)}\phi_1\overline{\phi_2}.$$

Adding the complex conjugates of the second of these two equations to the first yields

$$|u_1|(|\phi_1|^2+|\phi_2|^2) = e^{-i\alpha}\psi_1\overline{\phi_1} - |u_2|e^{i(\beta-\alpha)}\phi_2\overline{\phi_1} + e^{i\gamma}\overline{\psi_2}\phi_2 - |u_2|e^{i(\gamma-\theta)}\overline{\phi_1}\phi_2.$$

Multiplying by $|u_1|$ we obtain

$$\begin{aligned}|u_1|^2(|\phi_1|^2+|\phi_2|^2) &= e^{-i\alpha}|u_1|\left(\psi_1\overline{\phi_1}+e^{i(\gamma+\alpha)}\overline{\psi_2}\phi_2\right) - |u_1||u_2|\left(e^{i(\beta-\alpha)}+e^{i(\gamma-\theta)}\right)\phi_2\overline{\phi_1}.\\ &= e^{-i\alpha}|u_1|\left(\psi_1\overline{\phi_1}+e^{i(\gamma+\alpha)}\overline{\psi_2}\phi_2\right).\end{aligned}$$

Thus we have the two solutions $u_1 = 0$ or

$$u_1 = |u_1|e^{i\alpha} = \frac{\psi_1\overline{\phi_1}+e^{i(\gamma+\alpha)}\overline{\psi_2}\phi_2}{|\phi_1|^2+|\phi_2|^2}.$$

It can be verified that when $u_1 = 0$, $\psi_1 = u_2\phi_2$, $\psi_2 = u_3\phi_1$ the above solution also yields $u_1 = 0$. Similarly we solve for u_2, u_3 and u_4 to obtain the solution

$$u_1 = \frac{\psi_1\overline{\phi_1}+e^{i(\gamma+\alpha)}\overline{\psi_2}\phi_2}{|\phi_1|^2+|\phi_2|^2}, \qquad u_2 = \frac{\psi_1\overline{\phi_2}+e^{i(\theta+\beta)}\overline{\psi_2}\phi_1}{|\phi_2|^2+|\phi_1|^2},$$

$$u_3 = \frac{\psi_2\overline{\phi_1}+e^{i(\theta+\beta)}\overline{\psi_1}\phi_2}{|\phi_1|^2+|\phi_2|^2}, \qquad u_4 = \frac{\psi_2\overline{\phi_2}+e^{i(\gamma+\alpha)}\overline{\psi_1}\phi_1}{|\phi_1|^2+|\phi_2|^2}.$$

Assuming $u_1, u_2 \neq 0$ equation (3) yields

$$e^{i(\alpha+\gamma)} = -e^{i(\beta+\theta)}.$$

Thus the solution can be expressed as

$$u_1 = \frac{\psi_1\overline{\phi_1}+e^{i\mu}\overline{\psi_2}\phi_2}{|\phi_1|^2+|\phi_2|^2}, \qquad u_2 = \frac{\psi_1\overline{\phi_2}-e^{i\mu}\overline{\psi_2}\phi_1}{|\phi_2|^2+|\phi_1|^2},$$

$$u_3 = \frac{\psi_2\overline{\phi_1}-e^{i\mu}\overline{\psi_1}\phi_2}{|\phi_1|^2+|\phi_2|^2}, \qquad u_4 = \frac{\psi_2\overline{\phi_2}+e^{i\mu}\overline{\psi_1}\phi_1}{|\phi_1|^2+|\phi_2|^2}$$

where $\mu := \alpha+\gamma \in \mathbb{R}$ is arbitrary.

(b) Assuming $\|\,|\psi\rangle\,\| = \|\,|\phi\rangle\,\| \neq 0$ we know that $\{\,|\phi\rangle, |\phi_\perp\rangle\,\}$ forms an orthogonal basis for $\mathbb{C}^2$ and $\{\,|\psi\rangle, |\psi_\perp\rangle\,\}$ also forms an orthogonal basis for $\mathbb{C}^2$. We also know that

$$0 = \langle\phi|\phi_\perp\rangle = \langle\phi|U^*U|\phi_\perp\rangle = (U|\phi\rangle)^*U|\phi_\perp\rangle = \langle\psi|U|\phi_\perp\rangle.$$

Since an arbitrary state in $\mathbb{C}^2$ can be written as $\alpha|\phi\rangle + \beta|\phi_\perp\rangle$ we have

$$U(\alpha|\phi\rangle+\beta|\phi_\perp\rangle) = \alpha U|\phi\rangle + \beta U|\phi_\perp\rangle = \alpha|\psi\rangle + \beta U|\phi_\perp\rangle$$

where $U|\phi_\perp\rangle = \mu|\psi\rangle + \nu|\psi_\perp\rangle$. From $\langle\psi|U|\phi_\perp\rangle = 0$ we obtain $\mu = 0$ and $U|\phi_\perp\rangle = \nu|\psi_\perp\rangle$. It follows that

$$U = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\phi\rangle} + \nu\frac{|\psi_\perp\rangle\langle\phi_\perp|}{\langle\phi_\perp|\phi_\perp\rangle} = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\phi\rangle} + \nu|\psi_\perp\rangle\langle\phi_\perp|.$$

From $UU^* = I_2$ we find

$$\left(\frac{|\psi\rangle\langle\phi|}{\langle\phi|\phi\rangle} + \nu|\psi_\perp\rangle\langle\phi_\perp|\right)\left(\frac{|\phi\rangle\langle\psi|}{\langle\phi|\phi\rangle} + \overline{\nu}|\phi_\perp\rangle\langle\psi_\perp|\right) = \frac{|\psi\rangle\langle\psi|}{\langle\phi|\phi\rangle} + |\nu|^2|\psi_\perp\rangle\langle\psi_\perp| = I_2.$$

Now

$$UU^*|\psi_\perp\rangle = I_2|\psi_\perp\rangle = |\nu|^2|\psi_\perp\rangle = |\psi_\perp\rangle$$

so that $|\nu|^2 = 1$, i.e. ν has the form $\nu = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Thus we find

$$U = \frac{|\psi\rangle\langle\phi|}{\langle\phi|\phi\rangle} + e^{i\theta}|\psi_\perp\rangle\langle\phi_\perp|.$$

This is the general solution for $\langle\phi|\phi\rangle \neq 0$.

Note: Noting that we can use $|\phi_\perp\rangle = |\psi\rangle$ and $|\psi_\perp\rangle = |\phi\rangle$ for question 1. we find the two solutions given in 1. by choosing $\theta = 0$ and $\theta = \pi$.

3. Since

$$\mathbf{e}_{1,2}\mathbf{e}_{1,2}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}_{1,2}\mathbf{e}_{2,2}^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{e}_{2,2}\mathbf{e}_{1,2}^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e}_{2,2}\mathbf{e}_{2,2}^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we find

$$U_{QFT,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i2\pi/2} & e^{-i2\pi 2/2} \\ e^{-i2\pi 2/2} & e^{-i2\pi 4/2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi} & e^{-i2\pi} \\ e^{-i2\pi} & e^{-i4\pi} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since

$$\mathbf{e}_{1,4}\mathbf{e}_{1,4}^* = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{1,4}\mathbf{e}_{2,4}^* = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{1,4}\mathbf{e}_{3,4}^* = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{1,4}\mathbf{e}_{4,4}^* = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix},$$

$$\mathbf{e}_{2,4}\mathbf{e}_{1,4}^* = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{2,4}\mathbf{e}_{2,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{2,4}\mathbf{e}_{3,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{2,4}\mathbf{e}_{4,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix},$$

$$\mathbf{e}_{3,4}\mathbf{e}_{1,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{3,4}\mathbf{e}_{2,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{3,4}\mathbf{e}_{3,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix}, \ \mathbf{e}_{3,4}\mathbf{e}_{4,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix},$$

$$\mathbf{e}_{4,4}\mathbf{e}_{1,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}, \ \mathbf{e}_{4,4}\mathbf{e}_{2,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \end{pmatrix}, \ \mathbf{e}_{4,4}\mathbf{e}_{3,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix}, \ \mathbf{e}_{4,4}\mathbf{e}_{4,4}^* = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \end{pmatrix}$$

we find

$$U_{QFT,4} = \frac{1}{2} \begin{pmatrix} e^{-i2\pi/4} & e^{-i2\pi 2/4} & e^{-i2\pi 3/4} & e^{-i2\pi 4/4} \\ e^{-i2\pi 2/4} & e^{-i2\pi 4/4} & e^{-i2\pi 6/4} & e^{-i2\pi 8/4} \\ e^{-i2\pi 3/4} & e^{-i2\pi 6/4} & e^{-i2\pi 9/4} & e^{-i2\pi 12/4} \\ e^{-i2\pi 4/4} & e^{-i2\pi 8/4} & e^{-i2\pi 12/4} & e^{-i2\pi 16/4} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -i & -1 & i & 1 \\ -1 & 1 & -1 & 1 \\ i & -1 & -i & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Noting that

$$\mathbf{e}_{1,4} = \mathbf{e}_{1,2} \otimes \mathbf{e}_{1,2}, \quad \mathbf{e}_{2,4} = \mathbf{e}_{1,2} \otimes \mathbf{e}_{2,2}, \quad \mathbf{e}_{3,4} = \mathbf{e}_{2,2} \otimes \mathbf{e}_{1,2}, \quad \mathbf{e}_{4,4} = \mathbf{e}_{2,2} \otimes \mathbf{e}_{2,2}$$

we find that

$$\begin{aligned} U_{QFT,2} &= \frac{1}{2} \sum_{s=1}^{2} \sum_{j=1}^{2} \sum_{t=1}^{2} \sum_{k=1}^{2} e^{-i2\pi(2s+j)(2k+t)/4} (\mathbf{e}_{s,2} \otimes \mathbf{e}_{j,2})(\mathbf{e}_{k,2} \otimes \mathbf{e}_{t,2})^* \\ &= \frac{1}{2} \sum_{s=1}^{2} \sum_{j=1}^{2} \sum_{t=1}^{2} \sum_{k=1}^{2} e^{-i2\pi(4ts+2ks+2jt+jk)/4} (\mathbf{e}_{s,2}\mathbf{e}_{k,2}^*) \otimes (\mathbf{e}_{j,2}\mathbf{e}_{t,2}^*) \\ &= \sum_{j=1}^{2} \sum_{k=1}^{2} e^{-i2\pi jk/4} \left(\sum_{s=1}^{2} \frac{1}{\sqrt{2}} e^{-i2\pi ks/2} \mathbf{e}_{s,2}\mathbf{e}_{k,2}^* \right) \otimes \left(\sum_{t=1}^{2} \frac{1}{\sqrt{2}} e^{-i2\pi jt/2} \mathbf{e}_{j,2}\mathbf{e}_{t,2}^* \right) \end{aligned}$$

The terms in the Kronecker product also appear in $U_{QFT,1}$ and has a form similar to $U_{QFT,1} \otimes U_{QFT,1}$, however the extra term $e^{-i2\pi jk/4}$ means that $U_{QFT,2} \neq U_{QFT,1} \otimes U_{QFT,1}$. In fact, $U_{QFT,2}$ cannot be written as a Kronecker product of 2×2 matrices.