University of Johannesburg

Applied Mathematics 3B

Assignment #7 **Solutions**

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1. Since
$$\langle\psi|\psi\rangle = \cos^2\theta + \sin^2\theta = 1, \qquad \langle\phi|\phi\rangle = \cos^2\phi + \sin^2\phi = 1$$
$|\psi\rangle$ and $|\phi\rangle$ are normalized. From
$$\langle\psi|\phi\rangle = \cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta-\phi)$$
we find that $|\psi\rangle$ and $|\phi\rangle$ are not orthogonal in general (they are orthogonal when $\theta-\phi = (2k+1)\pi/2$ where $k\in\mathbb{Z}$). We find
$$\rho = \begin{pmatrix} p\cos^2\theta + (1-p)\cos^2\phi & p\cos\theta\sin\theta + (1-p)\cos\phi\sin\phi \\ p\sin\theta\cos\theta + (1-p)\sin\phi\cos\phi & p\sin^2\theta + (1-p)\sin^2\phi \end{pmatrix}.$$
It follows that
$$\mathrm{tr}\rho = p\cos^2\theta + (1-p)\cos^2\phi + p\sin^2\theta + (1-p)\sin^2\phi = p(\cos^2\theta+\sin^2\theta) + (1-p)(\cos^2\phi+\sin^2\phi) = 1.$$
From
$$\begin{aligned} |\psi\rangle\langle\psi|\rho &= p|\psi\rangle\langle\psi|\psi\rangle\langle\psi| + (1-p)|\psi\rangle\langle\psi|\phi\rangle\langle\phi| = p|\psi\rangle\langle\psi| + (1-p)\cos(\theta-\phi)|\psi\rangle\langle\phi| \\ &= \begin{pmatrix} p\cos^2\theta + (1-p)\cos(\theta-\phi)\cos\theta\cos\phi & p\cos\theta\sin\theta + (1-p)\cos(\theta-\phi)\cos\theta\sin\phi \\ p\sin\theta\cos\theta + (1-p)\cos(\theta-\phi)\sin\theta\cos\phi & p\sin^2\theta + (1-p)\cos(\theta-\phi)\sin\theta\cos\phi \end{pmatrix} \end{aligned}$$
it follows that
$$\mathrm{tr}(|\psi\rangle\langle\psi|\rho) = p\cos^2\theta + (1-p)\cos(\theta-\phi)\cos\theta\cos\phi + p\sin^2\theta + (1-p)\cos(\theta-\phi)\sin\theta\cos\phi = p + (1-p)\cos^2(\theta-\phi).$$
Suppose we prepare multiple systems in the state $|\psi\rangle$ and $|\phi\rangle$ in the ratio $p:(1-p)$. The average probability of finding (measuring) the systems in the state $|\psi\rangle$ is
$$p|\langle\psi|\psi\rangle|^2 + (1-p)|\langle\psi|\phi\rangle|^2 = p + (1-p)\cos^2(\theta-\phi) = \mathrm{tr}(|\psi\rangle\langle\psi|\rho)$$
where $|\langle\psi|\psi\rangle|^2 = 1$ is the probability of finding $|\psi\rangle$ in the state $|\psi\rangle$ and $|\langle\psi|\phi\rangle|^2 = \cos^2(\theta-\phi)$ is the average probability of finding $|\phi\rangle$ in the state $|\psi\rangle$.

To determine whether the eigenvalues of ρ are non-negative we could calculate the eigenvalues of ρ directly. However, let $|x\rangle$ be a normalized eigenvector of ρ corresponding to the eigenvalue λ (i.e. $\rho|x\rangle = \lambda|x\rangle$). Then we find
$$\langle x|\rho|x\rangle = \langle x|\lambda|x\rangle = \lambda\langle x|x\rangle = \lambda$$
since $\langle x|x\rangle = 1$. We also find
$$\langle x|\rho|x\rangle = p\langle x|\psi\rangle\langle\psi|x\rangle + (1-p)\langle x|\phi\rangle\langle\phi|x\rangle = p\langle x|\psi\rangle\overline{\langle x|\psi\rangle} + (1-p)\langle x|\phi\rangle\overline{\langle x|\phi\rangle} = p|\langle x|\psi\rangle|^2 + (1-p)|\langle x|\phi\rangle|^2 \geq 0$$
since $p\geq 0$ and $1-p\geq 0$. Thus $\lambda\geq 0$, i.e. the eigenvalues of ρ are non-negative.

Notice that the eigenvalue λ is the average probability of finding the systems in the eigenstate $|x\rangle$.

2. Since
$$|\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}, \qquad |\phi\rangle\langle\phi| = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\theta & \sin^2\phi \end{pmatrix}$$

we find

$$\rho = \begin{pmatrix} \cos^2\theta\cos^2\phi & \cos^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos^2\phi & \cos\theta\sin\theta\cos\phi\sin\phi \\ \cos^2\theta\sin\phi\cos\phi & \cos^2\theta\sin^2\theta & \cos\theta\sin\theta\sin\phi\cos\phi & \cos\theta\sin\theta\sin^2\phi \\ \sin\theta\cos\theta\cos^2\phi & \sin\theta\cos\theta\cos\phi\sin\phi & \sin^2\theta\cos^2\phi & \sin^2\theta\cos\phi\sin\phi \\ \sin\theta\cos\theta\sin\phi\cos\phi & \sin\theta\cos\theta\sin^2\phi & \sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi \end{pmatrix}.$$

It follows that

$$\mathrm{tr}\rho = \mathrm{tr}(|\psi\rangle\langle\psi|)\ \mathrm{tr}(|\phi\rangle\langle\phi|) = 1.$$

For ρ_1 we find

$$\begin{aligned} \rho_1 &= \sum_{j=1}^{2}(I_2\otimes\mathbf{e}_{j,2}^T)(|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|)(I_2\otimes\mathbf{e}_{j,2}) = \sum_{j=1}^{2}(I_2|\psi\rangle\langle\psi|I_2)\otimes(\mathbf{e}_{j,2}^T|\phi\rangle\langle\phi|\mathbf{e}_{j,2}) \\ &= \sum_{j=1}^{2}|\psi\rangle\langle\psi|\otimes(\mathbf{e}_{j,2}^T|\phi\rangle\langle\phi|\mathbf{e}_{j,2}) = |\psi\rangle\langle\psi|\otimes\sum_{j=1}^{2}(\mathbf{e}_{j,2}^T|\phi\rangle\langle\phi|\mathbf{e}_{j,2}) \\ &= |\psi\rangle\langle\psi|\otimes(\mathrm{tr}|\phi\rangle\langle\phi|) = |\psi\rangle\langle\psi|. \end{aligned}$$

Similarly

$$\rho_2 = (\mathrm{tr}|\psi\rangle\langle\psi|)\otimes|\phi\rangle\langle\phi| = |\phi\rangle\langle\phi|.$$

Obviously $\mathrm{tr}\rho_1 = \mathrm{tr}\rho_2 = 1$.

3. Noting that $\mathrm{tr}(|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|) = 1$ from question 2 and

$$\mathrm{tr}(|\phi\rangle\langle\phi|\otimes|\psi\rangle\langle\psi|) = \mathrm{tr}(|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|)\Big|_{\theta\to\phi,\phi\to\theta} = 1\Big|_{\theta\to\phi,\phi\to\theta} = 1$$

we have

$$\mathrm{tr}\rho = p\mathrm{tr}(|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|) + (1-p)\mathrm{tr}(|\phi\rangle\langle\phi|\otimes|\psi\rangle\langle\psi|) = p + (1-p) = 1.$$

For ρ_1 we find

$$\begin{aligned} \rho_1 &= p\sum_{j=1}^{2}(I_2\otimes\mathbf{e}_{j,2}^T)(|\psi\rangle\langle\psi|\otimes|\phi\rangle\langle\phi|)(I_2\otimes\mathbf{e}_{j,2}) + (1-p)\sum_{j=1}^{2}(I_2\otimes\mathbf{e}_{j,2}^T)(|\phi\rangle\langle\phi|\otimes|\psi\rangle\langle\psi|)(I_2\otimes\mathbf{e}_{j,2}) \\ &= p|\psi\rangle\langle\psi|\otimes(\mathrm{tr}|\phi\rangle\langle\phi|) + (1-p)|\phi\rangle\langle\phi|\otimes(\mathrm{tr}|\psi\rangle\langle\psi|) = p|\psi\rangle\langle\psi| + (1-p)|\phi\rangle\langle\phi|. \end{aligned}$$

Similarly

$$\rho_2 = p|\phi\rangle\langle\phi| + (1-p)|\psi\rangle\langle\psi|.$$

Thus

$$\mathrm{tr}\rho_1 = p\,\mathrm{tr}|\psi\rangle\langle\psi| + (1-p)\mathrm{tr}|\phi\rangle\langle\phi| = p + 1 - p = 1$$

and

$$\mathrm{tr}\rho_2 = p\,\mathrm{tr}|\phi\rangle\langle\phi| + (1-p)\mathrm{tr}|\psi\rangle\langle\psi| = p + 1 - p = 1.$$