University of Johannesburg

Applied Mathematics 3B

Assignment #6 **Solutions**

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1. We find

$$\mathrm{vec}_{2\times 2}A = \sum_{j=1}^{2}(A\mathbf{e}_{j,2})\otimes\mathbf{e}_{j,2} = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix},$$

$$\mathrm{vec}_{2\times 2}B = \sum_{j=1}^{2}(B\mathbf{e}_{j,2})\otimes\mathbf{e}_{j,2} = \begin{pmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{pmatrix}.$$

Notice that the entries of $\mathrm{vec}_{2\times 2}A$ are exactly the entries of A. We have

$$\mathrm{vec}_{4\times 4}A\otimes B = \sum_{j=1}^{4}((A\otimes B)\mathbf{e}_{j,4})\otimes\mathbf{e}_{j,4} = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{12} \\ a_{12}b_{11} \\ a_{12}b_{12} \\ a_{11}b_{21} \\ a_{11}b_{22} \\ a_{12}b_{21} \\ a_{12}b_{22} \\ a_{21}b_{11} \\ a_{21}b_{12} \\ a_{22}b_{11} \\ a_{22}b_{12} \\ a_{21}b_{21} \\ a_{21}b_{22} \\ a_{22}b_{21} \\ a_{22}b_{22} \end{pmatrix}.$$

Now we determine the two matrices

$$\sum_{j=1}^{2}\underbrace{\mathbf{e}_{j,2}}_{2\times 1}\otimes\underbrace{I_2}_{2\times 2}\otimes\underbrace{(A\mathbf{e}_{j,2})}_{2\times 1}\otimes\underbrace{I_2}_{2\times 2} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \\ 0 & 0 & a_{21} & 0 \\ 0 & 0 & 0 & a_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{12} & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 \\ a_{22} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & a_{12} \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{21} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \\ 0 & 0 & a_{21} & 0 \\ 0 & 0 & 0 & a_{21} \\ a_{12} & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 \\ a_{22} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{12} & 0 \\ 0 & 0 & 0 & a_{12} \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{22} \end{pmatrix}$$

and

$$\sum_{j=1}^{2} \underbrace{I_2}_{2\times 2} \otimes \underbrace{\mathbf{e}_{j,2}}_{2\times 1} \otimes \underbrace{I_2}_{2\times 2} \otimes \underbrace{(B\mathbf{e}_{j,2})}_{2\times 1} = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \\ 0 & b_{11} & 0 & 0 \\ 0 & b_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11} & 0 \\ 0 & 0 & b_{21} & 0 \\ 0 & 0 & 0 & b_{11} \\ 0 & 0 & 0 & b_{21} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{12} & 0 & 0 & 0 \\ b_{22} & 0 & 0 & 0 \\ 0 & b_{12} & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_{12} & 0 \\ 0 & 0 & b_{22} & 0 \\ 0 & 0 & 0 & b_{12} \\ 0 & 0 & 0 & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \\ 0 & b_{11} & 0 & 0 \\ 0 & b_{21} & 0 & 0 \\ b_{12} & 0 & 0 & 0 \\ b_{22} & 0 & 0 & 0 \\ 0 & b_{12} & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & b_{11} & 0 \\ 0 & 0 & b_{21} & 0 \\ 0 & 0 & 0 & b_{11} \\ 0 & 0 & 0 & b_{21} \\ 0 & 0 & b_{12} & 0 \\ 0 & 0 & b_{22} & 0 \\ 0 & 0 & 0 & b_{12} \\ 0 & 0 & 0 & b_{22} \end{pmatrix}.$$

The use of these matrices is as follows

$$\overbrace{\left(\sum_{j=1}^{2} \mathbf{e}_{j,2} \otimes I_2 \otimes (A\mathbf{e}_{j,2}) \otimes I_2\right)}^{A\otimes} \overbrace{\left(\sum_{j=1}^{2} \mathbf{e}_{j,2} \otimes (B\mathbf{e}_{j,2})\right)}^{B} = \overbrace{\sum_{j=1}^{4} \mathbf{e}_{j,4} \otimes ((A\otimes B)\mathbf{e}_{j,4})}^{A\otimes B},$$

$$\overbrace{\left(\sum_{j=1}^{2} I_2 \otimes \mathbf{e}_{j,2} \otimes I_2 \otimes (B\mathbf{e}_{j,2})\right)}^{\otimes B} \overbrace{\left(\sum_{j=1}^{2} \mathbf{e}_{j,2} \otimes (A\mathbf{e}_{j,2})\right)}^{A} = \overbrace{\sum_{j=1}^{4} \mathbf{e}_{j,4} \otimes ((A\otimes B)\mathbf{e}_{j,4})}^{A\otimes B}.$$

Thus we have expressed the Kronecker product as a matrix which operates on a vector (derived from the right hand matrix of the Kronecker product).