University of Johannesburg

Applied Mathematics 3B

Assignment #5 **Solutions**

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1. (a) We show that $|0'\rangle$ and $|1'\rangle$ are normalized and orthogonal:

$$\langle 0'|0'\rangle = \Big(\overline{a}\langle 0| + \overline{b}\langle 1|\Big)\Big(a|0\rangle + b|1\rangle\Big) = |a|^2\langle 0|0\rangle + \overline{a}b\langle 0|1\rangle + \overline{b}a\langle 1|0\rangle + |b|^2\langle 1|1\rangle = |a|^2 + |b|^2 = 1$$

$$\langle 1'|1'\rangle = \Big(a\langle 1| - b\langle 0|\Big)\Big(\overline{a}|1\rangle - \overline{b}|0\rangle\Big) = |a|^2\langle 1|1\rangle - b\overline{a}\langle 0|1\rangle - a\overline{b}\langle 1|0\rangle + |b|^2\langle 0|0\rangle = |a|^2 + |b|^2 = 1$$

$$\langle 0'|1'\rangle = \Big(\overline{a}\langle 0| + \overline{b}\langle 1|\Big)\Big(\overline{a}|1\rangle - \overline{b}|0\rangle\Big) = \overline{a}^2\langle 0|1\rangle + \overline{b}\overline{a}\langle 1|1\rangle - \overline{a}\overline{b}\langle 0|0\rangle - \overline{b}^2\langle 1|0\rangle = \overline{b}\overline{a} - \overline{a}\overline{b} = 0.$$

(b) We have

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}(|0'\rangle\otimes|1'\rangle - |1'\rangle\otimes|0'\rangle) \\
&= \frac{1}{\sqrt{2}}\Big((a|0\rangle + b|1\rangle)\otimes(\overline{a}|1\rangle - \overline{b}|0\rangle) - (\overline{a}|1\rangle - \overline{b}|0\rangle)\otimes(a|0\rangle + b|1\rangle)\Big) \\
&= \frac{1}{\sqrt{2}}\Big(|a|^2|0\rangle\otimes|1\rangle - a\overline{b}|0\rangle\otimes|0\rangle + b\overline{a}|1\rangle\otimes|1\rangle - |b|^2|1\rangle\otimes|0\rangle \\
&\quad -\big(|a|^2|1\rangle\otimes|0\rangle + \overline{a}b|1\rangle\otimes|1\rangle - \overline{b}a|0\rangle\otimes|0\rangle - |b|^2|0\rangle\otimes|1\rangle\big)\Big) \\
&= \frac{1}{\sqrt{2}}\Big((|a|^2 + |b|^2)|0\rangle\otimes|1\rangle - (|a|^2 + |b|^2)|1\rangle\otimes|0\rangle)\Big) \\
&= \frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle)
\end{aligned}$$

Notice that the result is independent of a and b. The state $|\psi\rangle$ has the same form independent of the chosen basis (up to a global phase $e^{i\theta}$, $\theta\in\mathbb{R}$).

(c) To show that A is observable it suffices to show that $A^* = A$. Using

$$(|0\rangle\langle 0|)^* = \langle 0|^*|0\rangle^* = |0\rangle\langle 0|, \quad (|0\rangle\langle 1|)^* = \langle 1|^*|0\rangle^* = |1\rangle\langle 0|,$$

$$(|1\rangle\langle 0|)^* = \langle 0|^*|1\rangle^* = |0\rangle\langle 1|, \quad (|1\rangle\langle 1|)^* = \langle 1|^*|1\rangle^* = |1\rangle\langle 1|$$

we find that

$$A^* = (\alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1)^* = (\alpha|0\rangle\langle 0|)^* + (\beta|1\rangle\langle 1|)^* = \overline{\alpha}|0\rangle\langle 0| + \overline{\beta}|1\rangle\langle 1| = \alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1| = A$$

since $\overline{\alpha} = \alpha$ for $\alpha\in\mathbb{R}$. Since we work in $\mathbb{C}^2$, A has two eigenvalues. From

$$A|0\rangle = (\alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1)|0\rangle = \alpha|0\rangle\langle 0|0\rangle + \beta|1\rangle\langle 1|0\rangle = \alpha|0\rangle$$

and

$$A|1\rangle = (\alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1)|1\rangle = \alpha|0\rangle\langle 0|1\rangle + \beta|1\rangle\langle 1|1\rangle = \beta|1\rangle$$

the two eigenvalues (i.e. measurement outcomes) are α and β with corresponding orthonormal eigenstates $|0\rangle$ and $|1\rangle$.

Thus for measuring the first qubit we consider the observable $A\otimes I_2$ with eigenvalues α (eigenstates $|0\rangle\otimes|0\rangle$ and $|0\rangle\otimes|1\rangle$) and β (eigenstates $|1\rangle\otimes|0\rangle$ and $|1\rangle\otimes|1\rangle$). This is not the only choice for the corresponding eigenstates, but is a convenient one.

It will be useful to calculate some scalar products in advance:

$$(|0\rangle\otimes|0\rangle)^*|\psi\rangle = (\langle 0|\otimes\langle 0|)\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle-|1\rangle\otimes|0\rangle) = \frac{1}{\sqrt{2}}(\langle 0|0\rangle\otimes\langle 0|1\rangle-\langle 0|1\rangle\otimes\langle 0|0\rangle) = \frac{1}{\sqrt{2}}(1\otimes 0-0\otimes 1) = 0$$

$$(|0\rangle\otimes|1\rangle)^*|\psi\rangle = (\langle 0|\otimes\langle 1|)\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle-|1\rangle\otimes|0\rangle) = \frac{1}{\sqrt{2}}(\langle 0|0\rangle\otimes\langle 1|1\rangle-\langle 0|1\rangle\otimes\langle 1|0\rangle) = \frac{1}{\sqrt{2}}(1\otimes 1-0\otimes 0) = \frac{1}{\sqrt{2}}$$

$$(|1\rangle\otimes|0\rangle)^*|\psi\rangle = (\langle 1|\otimes\langle 0|)\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle-|1\rangle\otimes|0\rangle) = \frac{1}{\sqrt{2}}(\langle 1|0\rangle\otimes\langle 0|1\rangle-\langle 1|1\rangle\otimes\langle 0|0\rangle) = \frac{1}{\sqrt{2}}(0\otimes 0-1\otimes 1) = -\frac{1}{\sqrt{2}}$$

$$(|1\rangle\otimes|1\rangle)^*|\psi\rangle = (\langle 1|\otimes\langle 1|)\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle-|1\rangle\otimes|0\rangle) = \frac{1}{\sqrt{2}}(\langle 1|0\rangle\otimes\langle 1|1\rangle-\langle 1|1\rangle\otimes\langle 1|0\rangle) = \frac{1}{\sqrt{2}}(0\otimes 1-1\otimes 0) = 0$$

We need to consider two possibilities, namely $\alpha = \beta$ and $\alpha \neq \beta$.

$\alpha = \beta$: There is only one measurement outcome: α. The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned}\Pi_\alpha &:= (|0\rangle\otimes|0\rangle)(|0\rangle\otimes|0\rangle)^* + (|0\rangle\otimes|1\rangle)(|0\rangle\otimes|1\rangle)^* + (|1\rangle\otimes|0\rangle)(|1\rangle\otimes|0\rangle)^* + (|1\rangle\otimes|1\rangle)(|1\rangle\otimes|1\rangle)^*\\ &= |0\rangle\langle 0|\otimes|0\rangle\langle 0| + |0\rangle\langle 0|\otimes|1\rangle\langle 1| + |1\rangle\langle 1|\otimes|0\rangle\langle 0| + |1\rangle\langle 1|\otimes|1\rangle\langle 1|\\ &= (|0\rangle\langle 0| + |1\rangle\langle 1|)\otimes(|0\rangle\langle 0| + |1\rangle\langle 1|)\\ &= I_2\otimes I_2 = I_4\end{aligned}$$

i.e. the identity operator. The probability of obtaining the measurement outcome α is

$$p_\alpha = \left|(|0\rangle\otimes|0\rangle)^*|\psi\rangle\right|^2 + \left|(|0\rangle\otimes|1\rangle)^*|\psi\rangle\right|^2 + \left|(|1\rangle\otimes|0\rangle)^*|\psi\rangle\right|^2 + \left|(|1\rangle\otimes|1\rangle)^*|\psi\rangle\right|^2 = 1.$$

Alternatively:

$$p_\alpha = \|\Pi_\alpha|\psi\rangle\|^2 = \||\psi\rangle\|^2 = \langle\psi|\psi\rangle = \frac{1}{\sqrt{2}}(\langle 0|\otimes\langle 0| - \langle 1|\otimes\langle 1|)|\psi\rangle = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$$

The state after measurement is the projected state

$$\Pi_\alpha|\psi\rangle = I_4|\psi\rangle = |\psi\rangle$$

which when normalized yields

$$|\phi\rangle = |\psi\rangle$$

since

$$\langle\psi|\psi\rangle = 1.$$

$\alpha \neq \beta$: **Measurement outcome α:** The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned}\Pi_\alpha &:= (|0\rangle\otimes|0\rangle)(|0\rangle\otimes|0\rangle)^* + (|0\rangle\otimes|1\rangle)(|0\rangle\otimes|1\rangle)^*\\ &= |0\rangle\langle 0|\otimes|0\rangle\langle 0| + |0\rangle\langle 0|\otimes|1\rangle\langle 1|\\ &= |0\rangle\langle 0|\otimes(|0\rangle\langle 0| + |1\rangle\langle 1|)\\ &= |0\rangle\langle 0|\otimes I_2.\end{aligned}$$

Note that

$$\Pi_\alpha|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 0|\otimes I_2)(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle\langle 0|0\rangle)\otimes|1\rangle - (|0\rangle\langle 0|1\rangle)\otimes|0\rangle = \frac{1}{\sqrt{2}}|0\rangle\otimes|1\rangle.$$

The probability of obtaining the measurement outcome α is

$$p_\alpha = \left|(|0\rangle\otimes|0\rangle)^*|\psi\rangle\right|^2 + \left|(|0\rangle\otimes|1\rangle)^*|\psi\rangle\right|^2 = \frac{1}{2}.$$

Alternatively:

$$p_\alpha = \|\Pi_\alpha|\psi\rangle\|^2 = (\Pi_\alpha|\psi\rangle)^*\Pi_\alpha|\psi\rangle = \frac{1}{\sqrt{2}}(\langle 0|\otimes\langle 1|)\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle) = \frac{1}{2}.$$

The state after measurement is the projected state

$$\Pi_\alpha|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |1\rangle$$

which when normalized yields

$$|\phi\rangle = \frac{\Pi_\alpha|\psi\rangle}{\|\Pi_\alpha|\psi\rangle\|} = \frac{\Pi_\alpha|\psi\rangle}{\sqrt{p_\alpha}} = |0\rangle \otimes |1\rangle.$$

Measurement outcome β: The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned}
\Pi_\beta &:= (|1\rangle \otimes |0\rangle)(|1\rangle \otimes |0\rangle)^* + (|1\rangle \otimes |1\rangle)(|1\rangle \otimes |1\rangle)^* \\
&= |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\
&= |1\rangle\langle 1| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&= |1\rangle\langle 1| \otimes I_2.
\end{aligned}$$

Note that

$$\Pi_\beta|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle\langle 1|\otimes I_2)(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle) = \frac{1}{\sqrt{2}}(|1\rangle\langle 1|0\rangle)\otimes|1\rangle - (|1\rangle\langle 1|1\rangle)\otimes|0\rangle = -\frac{1}{\sqrt{2}}|1\rangle\otimes|0\rangle.$$

The probability of obtaining the measurement outcome β is

$$p_\beta = \Big|(|1\rangle \otimes |0\rangle)^*|\psi\rangle\Big|^2 + \Big|(|1\rangle \otimes |1\rangle)^*|\psi\rangle\Big|^2 = \frac{1}{2}.$$

Alternatively:

$$p_\beta = \|\Pi_\beta|\psi\rangle\|^2 = (\Pi_\beta|\psi\rangle)^*\Pi_\alpha|\psi\rangle = (-1)\frac{1}{\sqrt{2}}(\langle 1| \otimes \langle 0|)(-1)\frac{1}{\sqrt{2}}(|1\rangle \otimes |0\rangle) = \frac{1}{2}.$$

The state after measurement is the projected state

$$\Pi_\beta|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle \otimes |0\rangle$$

which when normalized yields

$$|\phi\rangle = \frac{\Pi_\beta|\psi\rangle}{\|\Pi_\beta|\psi\rangle\|} = \frac{\Pi_\alpha|\psi\rangle}{\sqrt{p_\beta}} = -|1\rangle \otimes |0\rangle.$$

(d) **For measuring the second qubit we consider the observable** $I_2 \otimes A$ with eigenvalues α (eigenstates $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |0\rangle$) and β (eigenstates $|0\rangle \otimes |1\rangle$ and $|1\rangle \otimes |1\rangle$). This is not the only choice for the corresponding eigenstates, but is a convenient one.

The measurement of the second qubit depends on the results of the first measurement in (c). Thus we need to consider three cases $\alpha = \beta$, and the two outcomes α and β when $\alpha \neq \beta$.

$\alpha = \beta$: We have

$$|\phi\rangle = |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

There is only one measurement outcome: α. The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned}
\Pi_\alpha &:= (|0\rangle \otimes |0\rangle)(|0\rangle \otimes |0\rangle)^* + (|1\rangle \otimes |0\rangle)(|1\rangle \otimes |0\rangle)^* + (|0\rangle \otimes |1\rangle)(|0\rangle \otimes |1\rangle)^* + (|1\rangle \otimes |1\rangle)(|1\rangle \otimes |1\rangle)^* \\
&= |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |0\rangle\langle 0| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\
&= (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\
&= I_2 \otimes I_2 = I_4
\end{aligned}$$

i.e. the identity operator. As above, the probability of the measurement outcome α is $p_{\alpha,\alpha} = 1$.

$\alpha \neq \beta$: – **Measurement outcome for (c) was** α: We have

$$|\phi\rangle = |0\rangle \otimes |1\rangle.$$

Measurement outcome α**:** The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned} \Pi_{\alpha,\alpha} &:= (|0\rangle \otimes |0\rangle)(|0\rangle \otimes |0\rangle)^* + (|1\rangle \otimes |0\rangle)(|1\rangle \otimes |0\rangle)^* \\ &= I_2 \otimes |0\rangle\langle 0|. \end{aligned}$$

Note that

$$\Pi_\alpha |\phi\rangle = 0.$$

The probability of obtaining the measurement outcome α is $p_{\alpha,\alpha} = 0$.

Measurement outcome β**:** The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned} \Pi_{\alpha,\beta} &:= (|0\rangle \otimes |1\rangle)(|0\rangle \otimes |1\rangle)^* + (|1\rangle \otimes |1\rangle)(|1\rangle \otimes |1\rangle)^* \\ &= I_2 \otimes |1\rangle\langle 1|. \end{aligned}$$

Note that

$$\Pi_{\alpha,\beta} |\phi\rangle = |0\rangle \otimes |1\rangle = |\phi\rangle.$$

The probability of obtaining the measurement outcome β is $p_{\alpha,\beta} = 1$.

– **Measurement outcome for (c) was** β: We have

$$|\phi\rangle = -|1\rangle \otimes |0\rangle.$$

Measurement outcome α**:** The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned} \Pi_{\beta,\alpha} &:= (|0\rangle \otimes |0\rangle)(|0\rangle \otimes |0\rangle)^* + (|1\rangle \otimes |0\rangle)(|1\rangle \otimes |0\rangle)^* \\ &= I_2 \otimes |0\rangle\langle 0|. \end{aligned}$$

Note that

$$\Pi_\alpha |\phi\rangle = -|1\rangle \otimes |0\rangle = |\phi\rangle.$$

The probability of obtaining the measurement outcome α is $p_{\beta,\alpha} = 1$.

Measurement outcome β**:** The corresponding projection onto the eigenspace is determined from the eigenstates:

$$\begin{aligned} \Pi_{\beta,\beta} &:= (|0\rangle \otimes |1\rangle)(|0\rangle \otimes |1\rangle)^* + (|1\rangle \otimes |1\rangle)(|1\rangle \otimes |1\rangle)^* \\ &= I_2 \otimes |1\rangle\langle 1|. \end{aligned}$$

Note that

$$\Pi_{\beta,\beta} |\phi\rangle = 0.$$

The probability of obtaining the measurement outcome β is $p_{\beta,\beta} = 0$.

Tabulating the probabilities we find

	$\alpha = \beta$	$\alpha \neq b$			
Measurement outcomes	α, α	α, α	α, β	β, α	β, β
Probability	$p_\alpha p_{\alpha,\alpha} = 1$	$p_\alpha p_{\alpha,\alpha} = 0$	$p_\alpha p_{\alpha,\beta} = \frac{1}{2}$	$p_\beta p_{\beta,\alpha} = \frac{1}{2}$	$p_\beta p_{\beta,\beta} = 0$

Consequently, for $\alpha \neq \beta$, the probability that the two measurement outcomes are the same is 0 (impossible) and the probability that the two measurement outcomes are different is 1 (certain).

2. (a) We have

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

The eigenvalues are 0, 0 and 3 (so that the singular values are $\sqrt{3}$, 0 and 0) with corresponding orthonormal eigenvectors

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \qquad \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\-2\end{pmatrix}, \qquad \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

Thus we set

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}, \qquad \Sigma = (\,\sqrt{3} \quad 0 \quad 0\,), \qquad U = \frac{1}{\sqrt{3}}(\,1 \quad -1 \quad 1\,)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix} = (\,1\,).$$

The last two columns of V may be chosen differently. The singular value decomposition is not unique. Consequently

$$(\,1 \quad -1 \quad 1\,) = (\,1\,)(\,\sqrt{3} \quad 0 \quad 0\,)\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}^T.$$

(b) We have

$$\begin{pmatrix}1\\-1\\1\end{pmatrix}^T\begin{pmatrix}1\\-1\\1\end{pmatrix} = 3.$$

The eigenvalue is trivially 3 with the corresponding normalized eigenvector 1. The singular value is $\sqrt{3}$. Thus we find

$$V = (\,1\,), \qquad \Sigma = \begin{pmatrix}\sqrt{3}\\0\\0\end{pmatrix}, \qquad U = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

The first column of U is determined from

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}V$$

where V is a scalar. The remaining columns are chosen to ensure orthonormality (the choice is not unique). It follows that

$$\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}\begin{pmatrix}\sqrt{3}\\0\\0\end{pmatrix}(\,1\,)^T.$$

(c) We have

$$\begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\end{pmatrix}^T\begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\end{pmatrix} = \begin{pmatrix}2 & 1 & 1\\1 & 1 & 0\\1 & 0 & 1\end{pmatrix}$$

With eigenvalues 3, 1 and 0 and corresponding orthonormal eigenvectors

$$\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}, \qquad \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}.$$

The singular values are $\sqrt{3}$, 1 and 0. Thus we have

$$V = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix}, \qquad \Sigma = \begin{pmatrix}\sqrt{3} & 0 & 0\\0 & 1 & 0\end{pmatrix}, \qquad U = \begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}.$$

The columns of U follow from the first two columns of V

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\end{pmatrix}\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \frac{1}{1}\begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}.$$

It follows that

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix}^T .$$

3. It suffices to show that ϕ_1 and ϕ_2 are normalized and orthogonal.

$$\langle \phi_1, \phi_1 \rangle = \phi_1^T \phi_1 = \cos^2\theta + \sin^2\theta = 1$$

$$\langle \phi_2, \phi_2 \rangle = \phi_2^T \phi_2 = \sin^2\theta + \cos^2\theta = 1$$

$$\langle \phi_1, \phi_2 \rangle = \phi_1^T \phi_2 = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0.$$

We find

$$\begin{aligned} \sum_{j=1}^{2} \phi_j^T \phi_j &= (\cos\theta \quad \sin\theta) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + (-\sin\theta \quad \cos\theta) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \\ &= a_{11}\cos^2\theta + a_{12}\cos\theta\sin\theta + a_{21}\sin\theta\cos\theta + a_{22}\sin^2\theta \\ &+ a_{11}\sin^2\theta - a_{12}\sin\theta\cos\theta - a_{21}\cos\theta\sin\theta + a_{22}\cos^2\theta \\ &= a_{11} + a_{22} \end{aligned}$$

i.e. we have found the trace of A.