University of Johannesburg

Applied Mathematics 3B

Assignment #4 **Solutions**

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1. We have

$$\begin{aligned}
\mathbf{x}\otimes\mathbf{y} &= \begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}\otimes\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = \begin{pmatrix} x_1y_1\\ x_1y_2\\ x_1y_3\\ x_2y_1\\ x_2y_2\\ x_2y_3\\ x_3y_1\\ x_3y_2\\ x_3y_3\end{pmatrix}\\
\mathbf{x}^T\otimes\mathbf{y} &= \begin{pmatrix} x_1 & x_2 & x_3\end{pmatrix}\otimes\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = \begin{pmatrix} x_1y_1 & x_2y_2 & x_3y_3\\ x_1y_1 & x_2y_2 & x_3y_3\\ x_1y_1 & x_2y_2 & x_3y_3\end{pmatrix}\\
\mathbf{x}^T\mathbf{y} &= \begin{pmatrix} x_1 & x_2 & x_3\end{pmatrix}\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} = x_1y_1+x_2y_2+x_3y_3\\
\mathbf{x}\otimes\mathbf{y}^T &= \begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}\otimes\begin{pmatrix} y_1 & y_2 & y_3\end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3\\ x_2y_1 & x_2y_2 & x_2y_3\\ x_3y_1 & x_3y_2 & x_3y_3\end{pmatrix}\\
\mathbf{x}\mathbf{y}^T &= \begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}\begin{pmatrix} y_1 & y_2 & y_3\end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3\\ x_2y_1 & x_2y_2 & x_2y_3\\ x_3y_1 & x_3y_2 & x_3y_3\end{pmatrix} = \mathbf{x}\otimes\mathbf{y}^T.
\end{aligned}$$

2. (a) Since

$$\mathbf{a}\otimes\mathbf{b} = \begin{pmatrix} a_1b_1\\ a_1b_2\\ a_2b_1\\ a_2b_2\end{pmatrix}$$

we consider

$$\begin{aligned}
f(a_1,a_2,b_1,b_2) := \left\|\begin{pmatrix} 1\\ -1\\ -1\\ 1\end{pmatrix} - \mathbf{a}\otimes\mathbf{b}\right\|^2 &= (1-a_1b_1)^2+(-1-a_1b_2)^2+(-1-a_2b_1)^2+(1-a_2b_2)^2\\
&= (a_1b_1-1)^2+(a_1b_2+1)^2+(a_2b_1+1)^2+(a_2b_2-1)^2.
\end{aligned}$$

To minimize we must solve

$$\begin{aligned}
\frac{\partial f}{\partial a_1} &= 2(a_1b_1-1)b_1+2(a_1b_2+1)b_2=0\\
\frac{\partial f}{\partial a_2} &= 2(a_2b_1+1)b_1+2(a_2b_2-1)b_2=0\\
\frac{\partial f}{\partial b_1} &= 2(a_1b_1-1)a_1+2(a_2b_1+1)a_2=0\\
\frac{\partial f}{\partial b_2} &= 2(a_1b_2+1)a_1+2(a_2b_2-1)a_2=0
\end{aligned}$$

Adding the first two equations yields

$$2(a_1+a_2)(b_1^2+b_2^2)=0.$$

Adding the last two equations yields

$$2(b_1+b_2)(a_1^2+a_2^2)=0.$$

Since $b_1^2+b_2^2=0$ iff $b_1=b_2=0$ and $a_1^2+a_2^2=0$ iff $a_1=a_2=0$ we consider these cases first:

$$f(0,0,0,0)=f(a_1,a_2,0,0)=f(0,0,b_1,b_2)=4.$$

Now consider $a_1^2+a_2^2\neq 0$ (then $b_1=-b_2$) and $b_1^2+b_2^2\neq 0$ (then $a_1=-a_2$). The third equation then becomes

$$2(a_1b_1-1)a_1+2(-a_1b_1+1)(-a_1)=4a_1(a_1b_1-1)=0.$$

Since we have already considered $a_1=a_2=0$ we find $a_1b_1=1$. The extrema is then $a_2=-a_1$, $b_1=1/a_1$, $b_2=-1/a_1$ with

$$f\left(a_1,-a_1,\frac{1}{a_1},-\frac{1}{a_1}\right)=0.$$

Since $f(a_1,a_2,b_1,b_2)\geq 0$ this must be the global minimum:

$$\mathbf{a}=\begin{pmatrix}t\\-t\end{pmatrix},\qquad \mathbf{b}=\begin{pmatrix}\frac{1}{t}\\-\frac{1}{t}\end{pmatrix},\qquad t\in\mathbb{R}/\{0\}.$$

(b) Since

$$\mathbf{a}\otimes\mathbf{b}=\begin{pmatrix}a_1b_1\\a_1b_2\\a_2b_1\\a_2b_2\end{pmatrix}$$

we consider

$$\begin{aligned}g(a_1,a_2,b_1,b_2):=\left\|\begin{pmatrix}1\\0\\0\\1\end{pmatrix}-\mathbf{a}\otimes\mathbf{b}\right\|^2 &= (1-a_1b_1)^2+(-a_1b_2)^2+(-a_2b_1)^2+(1-a_2b_2)^2\\ &= (a_1b_1-1)^2+(a_1b_2)^2+(a_2b_1)^2+(a_2b_2-1)^2.\end{aligned}$$

To minimize we must solve

$$\begin{aligned}\frac{\partial g}{\partial a_1} &= 2(a_1b_1-1)b_1+2a_1b_2^2=0\\ \frac{\partial g}{\partial a_2} &= 2a_2b_1^2+2(a_2b_2-1)b_2=0\\ \frac{\partial g}{\partial b_1} &= 2(a_1b_1-1)a_1+2a_2^2b_1=0\\ \frac{\partial g}{\partial b_2} &= 2a_1^2b_2+2(a_2b_2-1)a_2=0\end{aligned}$$

These equations can be rewritten as

$$b_1=a_1(b_1^2+b_2^2),\qquad b_2=a_2(b_1^2+b_2^2),\qquad a_1=b_1(a_1^2+a_2^2),\qquad a_2=b_2(a_1^2+a_2^2)$$

If $b_1^2+b_2^2=0$ then $a_1=a_2=b_1=b_2=0$ and $g(0,0,0,0)=2$. Now consider $b_1^2+b_2^2\neq 0$. It follows that

$$a_1=\frac{b_1}{b_1^2+b_2^2},\qquad a_2=\frac{b_2}{b_1^2+b_2^2}.$$

For these values we find

$$g\left(\frac{b_1}{b_1^2+b_2^2},\frac{b_2}{b_1^2+b_2^2},b_1,b_2\right)=1.$$

To determine whether this is the global minimum we first note that

$$g(a_1,a_2,b_1,b_2)=(\mathbf{a}^T\mathbf{a})(\mathbf{b}^T\mathbf{b})-2\mathbf{a}^T\mathbf{b}+2.$$

The Cauchy-Schwarz inequality provides

$$(\mathbf{a}^T\mathbf{a})(\mathbf{b}^T\mathbf{b}) \geq (\mathbf{a}^T\mathbf{b})^2$$

so that

$$g(a_1,a_2,b_1,b_2) \geq (\mathbf{a}^T\mathbf{b})^2 - 2\mathbf{a}^T\mathbf{b} + 2 = (\mathbf{a}^T\mathbf{b} - 1)^2 + 1 \geq 1.$$

Thus the global minimum is found at

$$\mathbf{a} = \frac{1}{b_1^2+b_2^2}\mathbf{b}, \qquad \mathbf{b} \in \mathbb{R}^2/\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}.$$

3. (a)

$$\begin{aligned}
\sum_{j=1}^{4} \mathbf{e}_{j,4}^T A \mathbf{e}_{j,4} &= \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\\
&+ \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\\
&+ \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\
&+ \begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\\
&= a_{11} + a_{22} + a_{33} + a_{44} = \mathrm{tr}A.
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{j=1}^{2} (\mathbf{e}_{j,2} \otimes I_2)^T A (\mathbf{e}_{j,2} \otimes I_2) &= \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 0\end{pmatrix}\\
&+ \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} + \begin{pmatrix}a_{33} & a_{34}\\ a_{43} & a_{44}\end{pmatrix} = \begin{pmatrix}a_{11}+a_{33} & a_{12}+a_{34}\\ a_{21}+a_{43} & a_{22}+a_{44}\end{pmatrix}.
\end{aligned}$$

It follows that

$$\mathrm{tr}\left(\sum_{j=1}^{2} (\mathbf{e}_{j,2} \otimes I_2)^T A (\mathbf{e}_{j,2} \otimes I_2)\right) = \mathrm{tr}A.$$

We have performed a partial trace on A.