University of Johannesburg

Applied Mathematics 3B

Assignment #3 **Solutions**

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1. The Taylor series for $A(t)$ around $t = 0$ is

$$A(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left[\frac{dA(t)}{dt}\right]_{t=0}.$$

We have for $n > 0$

$$\frac{d^{n-1}}{dt^{n-1}} \left(-i\hbar \frac{dA(t)}{dt}\right) = \frac{d^{n-1}}{dt^{n-1}} \left([\hat{H}, A(t)]\right)$$

i.e.

$$\begin{aligned} -i\hbar \frac{d^n A(t)}{dt^n} &= \frac{d^{n-1}}{dt^{n-1}} \left(\hat{H}A(t) - A(t)\hat{H}\right) \\ &= \left(\hat{H} \frac{d^{n-1}A(t)}{dt^{n-1}} - \frac{d^{n-1}A(t)}{dt^{n-1}} \hat{H}\right) \\ &= \left[\hat{H}, \frac{d^{n-1}A(t)}{dt^{n-1}}\right] \end{aligned}$$

where we made use of the fact that $\hat{H}$ does not depend on t. It follows that

$$\frac{d^n A(t)}{dt^n} = \frac{i}{\hbar} \left[\hat{H}, \frac{d^{n-1}A(t)}{dt^{n-1}}\right].$$

Since

$$\frac{d^0 A(t)}{dt^0} = A(t) = [\hat{H}, A(t)]_0$$

it is clear that

$$\frac{d^n A(t)}{dt^n} = \frac{i}{\hbar} \left[\hat{H}, \frac{d^{n-1}A(t)}{dt^{n-1}}\right] = \frac{i}{\hbar} \left[\hat{H}, \frac{i}{\hbar} \left[\hat{H}, \frac{d^{n-1}A(t)}{dt^{n-1}}\right]\right] = \cdots = \left(\frac{i}{\hbar}\right)^n \left[\hat{H}, A(t)\right]_n.$$

At $t = 0$ we find

$$\left[\frac{d^n A(t)}{dt^n}\right]_{t=0} = \left(\frac{i}{\hbar}\right)^n \left[\hat{H}, A(0)\right]_n$$

since $\hat{H}$ is independent of t. Consequently

$$A(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left[\frac{dA(t)}{dt}\right]_{t=0} = \sum_{j=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^j}{j!} \left[\hat{H}, A(0)\right]_j.$$

2. (a) Since

$$\begin{aligned} [\hat{H}, A(0)]_0 &= A(0) = \sigma_x \\ [\hat{H}, A(0)]_1 &= [I_2, \sigma_x] = I_2\sigma_x - \sigma_x I_2 = \sigma_x - \sigma_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ [\hat{H}, A(0)]_2 &= \left[I_2, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = [\hat{H}, A(0)]_1 \end{aligned}$$

it follows that $[\hat{H}, A(0)]_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $n > 0$. Thus we find

$$A(t) = \sum_{j=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^j}{j!} \left[\hat{H}, A(0)\right]_j = A(0) + \sum_{j=1}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^j}{j!} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A(0) = \sigma_x.$$

We find that $A(t)$ does not change with time which is consistent with the fact that the derivative is the 2×2 zero matrix.

(b) Since

$$\begin{aligned}
[\hat{H}, A(0)]_0 &= A(0) = \sigma_x \\
[\hat{H}, A(0)]_1 &= [\sigma_x, \sigma_x] = \sigma_x\sigma_x - \sigma_x\sigma_x = \sigma_x - \sigma_x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
[\hat{H}, A(0)]_2 &= \left[\sigma_x, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = [\hat{H}, A(0)]_1
\end{aligned}$$

it follows that $[\hat{H}, A(0)]_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $n > 0$. Thus we find

$$A(t) = \sum_{j=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^j}{j!} \left[\hat{H}, A(0)\right]_j = A(0) + \sum_{j=1}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^j}{j!} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A(0) = \sigma_x.$$

We find that $A(t)$ does not change with time which is consistent with the fact that the derivative is the 2×2 zero matrix.

(c) Since

$$\begin{aligned}
[\hat{H}, A(0)]_0 &= A(0) = \sigma_x \\
[\hat{H}, A(0)]_1 &= [\sigma_z, \sigma_x] = \sigma_z\sigma_x - \sigma_x\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \\
[\hat{H}, A(0)]_2 &= \left[\sigma_z, \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \\
&= 4\sigma_x = 4[\hat{H}, A(0)]_0 \\
[\hat{H}, A(0)]_3 &= [\sigma_z, 4\sigma_x] = 4[\hat{H}, A(0)]_1
\end{aligned}$$

it follows that $[\hat{H}, A(0)]_{n+2} = 4[\hat{H}, A(0)]_n$. Thus we find

$$[\hat{H}, A(0)]_{2k} = 4^k\sigma_x = 2^{2k}\sigma_x,$$

$$[\hat{H}, A(0)]_{2k+1} = 4^k \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2^{2k+1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\begin{aligned}
A(t) &= \sum_{j=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^j}{j!} \left[\hat{H}, A(0)\right]_j = \sum_{k=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^{2k}}{(2k)!} \left[\hat{H}, A(0)\right]_{2k} = \sum_{k=0}^{\infty} \frac{\left(\frac{it}{\hbar}\right)^{2k+1}}{(2k+1)!} \left[\hat{H}, A(0)\right]_{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{\left(\frac{2it}{\hbar}\right)^{2k}}{(2k)!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{\left(\frac{2it}{\hbar}\right)^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \cosh\left(\frac{2it}{\hbar}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sinh\left(\frac{2it}{\hbar}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{\frac{2it}{\hbar}} \\ e^{-\frac{2it}{\hbar}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cos\left(\frac{2t}{\hbar}\right) + i\sin\left(\frac{2t}{\hbar}\right) \\ \cos\left(\frac{2t}{\hbar}\right) - i\sin\left(\frac{2t}{\hbar}\right) & 0 \end{pmatrix}.
\end{aligned}$$

3. From the definition of $\hat{q}$ we find that the eigenvalues of $\hat{q}$ are all x. Consider the eigenvalue equation for $\hat{p}$

$$\hat{p}f = \lambda f \qquad \Rightarrow \qquad -i\hbar\frac{df}{dx} = \lambda f.$$

Solving for $f \neq 0$ yields the eigenfunction corresponding to the eigenvalue λ. The solution is given by

$$f = e^{\frac{i}{\hbar}\int \lambda\, dx}[f]_{x=0} = e^{\frac{i}{\hbar}\lambda x}[f]_{x=0}$$

where we assumed λ is a constant with respect to x. Solving the equation does not place any constraints on λ[1]. Thus the eigenvalues of $\hat{p}$ are $\lambda \in \mathbb{C}$.

For the commutator we find

$$[\hat{p}, \hat{q}]f = \hat{p}(\hat{q}f) - \hat{q}(\hat{p}f) = -i\hbar \frac{d}{dx}(xf) - x\left(-i\hbar \frac{df}{dx}\right) = -i\hbar\left(f + x\frac{df}{dx}\right) + i\hbar\left(x\frac{df}{dx}\right) = -i\hbar f.$$

Thus $[\hat{p}, \hat{q}] = -i\hbar I$ where I is the appropriate identity operator $If := f$.

[1]**Note:** Requiring that f should be a member of a Hilbert space leads to constraints on λ, i.e. that λ should be real or even that the permissable values for λ be discrete. For example, working in the Hilbert space $L_2(\mathbb{R})$ where

$$\langle f(x), g(x)\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

we have

$$\begin{aligned}\langle f(x), f(x)\rangle &= \int_{-\infty}^{\infty} e^{\frac{i\lambda x}{\hbar}} f(0)\, \overline{e^{\frac{i\lambda x}{\hbar}} f(0)}\, dx = \int_{-\infty}^{\infty} e^{\frac{i\lambda x}{\hbar}} f(0)\, e^{\frac{-i\overline{\lambda} x}{\hbar}}\, \overline{f(0)}\, dx \\ &= \int_{-\infty}^{\infty} e^{\frac{i(\lambda-\overline{\lambda})x}{\hbar}} |f(0)|^2\, dx = \int_{-\infty}^{\infty} e^{-\frac{2\Im(\lambda)x}{\hbar}} |f(0)|^2\, dx\end{aligned}$$

where $\Im(\lambda)$ is the imaginary part of λ and x is real. The integral is not finite and f is not an element of the Hilbert space for any λ. On the other hand, considering non-constant $\lambda(x) = ix$ does yield an f that is an element of the Hilbert space.