University of Johannesburg

Applied Mathematics 3B

Assignment #2 **Solutions**

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1. We have
$$\mathbf{x}\mathbf{x}^* = \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} \overline{a}a & \overline{a}b \\ \overline{b}a & \overline{b}b \end{pmatrix} = \begin{pmatrix} |a|^2 & \overline{a}b \\ \overline{b}a & |b|^2 \end{pmatrix}.$$
Consequently
$$\mathrm{tr}(\mathbf{x}\mathbf{x}^*) = |a|^2 + |b|^2 = 1.$$
The eigenvalues of $\mathbf{x}\mathbf{x}^*$ are determined from the characteristic equation
$$\det(\lambda I_2 - \mathbf{x}\mathbf{x}^*) = \det \begin{pmatrix} \lambda - |a|^2 & -\overline{a}b \\ -\overline{b}a & \lambda - |b|^2 \end{pmatrix} = \lambda^2 - (|a|^2 + |b|^2)\lambda = \lambda^2 - \lambda = 0.$$
The solutions are $\lambda_1 = 1$ and $\lambda_2 = 0$. Thus the two eigenvalues are non-negative.

2. We must show that the set is linearly independent and that any 2×2 matrix over $\mathbb{C}$ can be represented as a linear combination of these matrices. To show linear independence we solve
$$a\frac{I_2}{\sqrt{2}} + b\frac{\sigma_x}{\sqrt{2}} + c\frac{\sigma_y}{\sqrt{2}} + d\frac{\sigma_z}{\sqrt{2}} = 0$$
for $a, b, c, d \in \mathbb{C}$. Multiplying the above equation by $\sqrt{2}$ we find
$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$$
i.e.
$$\begin{aligned} a + d &= 0 \\ b - ci &= 0 \\ b + ci &= 0 \\ a - d &= 0 \end{aligned}$$
with the only solution $a = b = c = d = 0$. Thus the set is linearly independent. Next we solve the matrix equation
$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \alpha\frac{I_2}{\sqrt{2}} + \beta\frac{\sigma_x}{\sqrt{2}} + \gamma\frac{\sigma_y}{\sqrt{2}} + \epsilon\frac{\sigma_z}{\sqrt{2}}$$
for $\alpha, \beta, \gamma, \epsilon \in \mathbb{C}$ in terms of $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Thus we obtain the equations
$$\begin{aligned} a_1 &= \frac{\alpha + \epsilon}{\sqrt{2}} \\ a_2 &= \frac{\beta - i\gamma}{\sqrt{2}} \\ a_3 &= \frac{\beta + i\gamma}{\sqrt{2}} \\ a_4 &= \frac{\alpha - \epsilon}{\sqrt{2}} \end{aligned}$$
with the solution
$$\alpha = \frac{a_1 + a_4}{\sqrt{2}}, \qquad \beta = \frac{a_2 + a_3}{\sqrt{2}}, \qquad \gamma = i\frac{a_2 - a_3}{\sqrt{2}}, \qquad \epsilon = \frac{a_1 - a_4}{\sqrt{2}}.$$

To show that we have an orthonormal basis we first note that

$$\left(\frac{I_2}{\sqrt{2}}\right)^2 = \left(\frac{\sigma_x}{\sqrt{2}}\right)^2 = \left(\frac{\sigma_y}{\sqrt{2}}\right)^2 = \left(\frac{\sigma_z}{\sqrt{2}}\right)^2 = \frac{1}{2}I_2.$$

$$\left(\frac{I_2}{\sqrt{2}}\right)^* = \frac{I_2}{\sqrt{2}}, \qquad \left(\frac{\sigma_x}{\sqrt{2}}\right)^* = \frac{\sigma_x}{\sqrt{2}}, \qquad \left(\frac{\sigma_y}{\sqrt{2}}\right)^* = \frac{\sigma_y}{\sqrt{2}}, \qquad \left(\frac{\sigma_z}{\sqrt{2}}\right)^* = \frac{\sigma_z}{\sqrt{2}}.$$

The scalar products are

$$\left\langle \frac{I_2}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr} I_2 = 1, \quad \left\langle \frac{I_2}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_x = 0, \quad \left\langle \frac{I_2}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_y = 0, \quad \left\langle \frac{I_2}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_z = 0$$

$$\left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_x = 0, \quad \left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr} I_2 = 1, \quad \left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_z = 0, \quad \left\langle \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = -\frac{1}{2}\operatorname{tr}\sigma_y = 0$$

$$\left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_y = 0, \quad \left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = -\frac{1}{2}\operatorname{tr}\sigma_z = 0, \quad \left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr} I_2 = 1, \quad \left\langle \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_x = 0$$

$$\left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{I_2}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_z = 0, \quad \left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr}\sigma_y = 0, \quad \left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}} \right\rangle = -\frac{1}{2}\operatorname{tr}\sigma_x = 0, \quad \left\langle \frac{\sigma_z}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}} \right\rangle = \frac{1}{2}\operatorname{tr} I_2 = 1$$

Thus the basis is orthonormal. Expanding the matrix we find

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{I_2}{\sqrt{2}} \right\rangle \frac{I_2}{\sqrt{2}} + \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{\sigma_x}{\sqrt{2}} \right\rangle \frac{\sigma_x}{\sqrt{2}} + \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{\sigma_y}{\sqrt{2}} \right\rangle \frac{\sigma_y}{\sqrt{2}} + \left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{\sigma_z}{\sqrt{2}} \right\rangle \frac{\sigma_z}{\sqrt{2}} \\
&= \left(\frac{a+d}{\sqrt{2}}\right)\frac{I_2}{\sqrt{2}} + \left(\frac{b+c}{\sqrt{2}}\right)\frac{\sigma_x}{\sqrt{2}} + \left(i\frac{b-c}{\sqrt{2}}\right)\frac{\sigma_y}{\sqrt{2}} + \left(\frac{a-d}{\sqrt{2}}\right)\frac{\sigma_z}{\sqrt{2}}.
\end{aligned}$$

Notice that this result coincides exactly with the solution found when we showed that the set forms a basis.

3. Since $A^0 = I_2$, $A^1 = A$ and $A^2 = I_2$, we find

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{A^j}{j!} &= \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(A^2)^k}{(2k)!} + \sum_{k=0}^{\infty} \frac{A(A^2)^k}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} \frac{I_2^k}{(2k)!} + \sum_{k=0}^{\infty} \frac{AI_2^k}{(2k+1)!} = I_2 \sum_{k=0}^{\infty} \frac{1}{(2k)!} + A \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \\
&= I_2 \cosh 1 + A \sinh 1 = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}.
\end{aligned}$$