University of Johannesburg

Applied Mathematics 3B

Assignment #1 **Solutions**

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1. Since the trace of a matrix is the sum of its eigenvalues and the determinant of a matrix is the product of its eigenvalues we find

$$\lambda - 1 + \lambda_2 = \mathrm{tr}A = 0, \qquad \lambda_1\lambda_2 = \det A = -2.$$

It follows that $\lambda_1 = \sqrt{2}$ and $\lambda_2 = -\sqrt{2}$ (or vice versa). Alternatively setting the characteristic polynomial $\det(\lambda I_2 - A) = (\lambda-1)(\lambda+1) - 1 = \lambda^2 - 2$ to zero yields $\lambda = \pm\sqrt{2}$. For the eigenvalue $\lambda_1 = \sqrt{2}$ we determine x and y from

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e.

$$x + y = \sqrt{2}x, \qquad x - y = \sqrt{2}y.$$

Consequently $y = (\sqrt{2}-1)x$. Thus the eigenspace corresponding to the eigenvalue $\sqrt{2}$ is

$$\begin{pmatrix} t \\ (\sqrt{2}-1)t \end{pmatrix}, \qquad t \in \mathbb{R}/\{0\}.$$

Normalizing we find

$$\mathbf{x}_1 = \pm\begin{pmatrix} \dfrac{1}{\sqrt{4-2\sqrt{2}}} \\ \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}.$$

For the eigenvalue $\lambda_2 = -\sqrt{2}$ we determine x and y from

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\sqrt{2}\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e.

$$x + y = -\sqrt{2}x, \qquad x - y = -\sqrt{2}y.$$

Consequently $y = (-\sqrt{2}-1)x$. Thus the eigenspace corresponding to the eigenvalue $-\sqrt{2}$ is

$$\begin{pmatrix} t \\ (-\sqrt{2}-1)t \end{pmatrix}, \qquad t \in \mathbb{R}/\{0\}.$$

Normalizing we find

$$\mathbf{x}_2 = \pm\begin{pmatrix} \dfrac{1}{\sqrt{4+2\sqrt{2}}} \\ \dfrac{-\sqrt{2}-1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}.$$

Now

$$\begin{aligned} \lambda_1\mathbf{x}_1^T\mathbf{x} + \lambda_2\mathbf{x}_2^T\mathbf{x}_2 &= \sqrt{2}\begin{pmatrix} \dfrac{1}{\sqrt{4-2\sqrt{2}}} \\ \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix}\begin{pmatrix} \dfrac{1}{\sqrt{4-2\sqrt{2}}} & \dfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \end{pmatrix} \\ &- \sqrt{2}\begin{pmatrix} \dfrac{1}{\sqrt{4+2\sqrt{2}}} \\ \dfrac{-\sqrt{2}-1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix}\begin{pmatrix} \dfrac{1}{\sqrt{4+2\sqrt{2}}} & \dfrac{-\sqrt{2}-1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{2}\begin{pmatrix} \frac{1}{4-2\sqrt{2}} & \frac{\sqrt{2}-1}{4-2\sqrt{2}} \\ \frac{\sqrt{2}-1}{4-2\sqrt{2}} & \frac{3-2\sqrt{2}}{4-2\sqrt{2}} \end{pmatrix} - \sqrt{2}\begin{pmatrix} \frac{1}{4+2\sqrt{2}} & \frac{-\sqrt{2}-1}{4+2\sqrt{2}} \\ \frac{-\sqrt{2}-1}{4+2\sqrt{2}} & \frac{3+2\sqrt{2}}{4+2\sqrt{2}} \end{pmatrix} \\
&= \frac{\sqrt{2}}{8}\left[\begin{pmatrix} 4+2\sqrt{2} & 2\sqrt{2} \\ 2\sqrt{2} & 4-2\sqrt{2} \end{pmatrix} - \begin{pmatrix} 4-2\sqrt{2} & -2\sqrt{2} \\ -2\sqrt{2} & 4+2\sqrt{2} \end{pmatrix}\right] \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = A.
\end{aligned}
$$

This is the expected result since A is hermitian.

2. The characteristic equation is

$$\lambda^2 - 1 = 0$$

i.e. $\lambda_1 = 1$ and $\lambda_2 = -1$. From the eigenvalue equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

we find $a = b$, i.e. we find a corresponding normalized eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

From the eigenvalue equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = -1 \cdot \begin{pmatrix} c \\ d \end{pmatrix}$$

we find $c = -d$, i.e. we find a corresponding normalized eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Consequently

$$1 \cdot \mathbf{x}_1\mathbf{x}_1^T + (-1) \cdot \mathbf{x}_2\mathbf{x}_2^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A.$$

3. Let the matrix be

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}.$$

The eigenvalue equations are

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The first equation yields $a_1 = 0$ and $a_2 = 0$ while the second equation yields $a_2 = 0$ and $a_4 = 1$. Thus we find the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Alternatively we construct a hermitian matrix with the desired properties from the eigenvalues and corresponding orthonormal eigenvectors

$$0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

4. Examples include

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \right\} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \right\}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \right\} \qquad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \right\}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \right\}$$

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ \sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \sqrt{\frac{1}{8}} \\ \sqrt{\frac{1}{8}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \right\}$$

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{pmatrix}, \right\} \qquad \theta \in \mathbb{R}$$