University of Johannesburg

Applied Mathematics 3B

Assignment #10 **Solutions**

---

1. We have

$$
\begin{aligned}
(U_{QFT,4} \otimes I_2)|\psi_1\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle) \otimes |0\rangle + (U_{QFT,4}|1\rangle) \otimes |1\rangle + (U_{QFT,4}|2\rangle) \otimes |0\rangle + (U_{QFT,4}|3\rangle) \otimes |1\rangle) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right) \otimes |0\rangle + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right) \otimes |1\rangle\right. \\
&\quad \left. + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right) \otimes |0\rangle + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right) \otimes |1\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right) \otimes |0\rangle + \left(\sum_{j=0}^{3} e^{-i2\pi j/4}|j\rangle\right) \otimes |1\rangle\right. \\
&\quad \left. + \left(\sum_{j=0}^{3} e^{-i4\pi j/4}|j\rangle\right) \otimes |0\rangle + \left(\sum_{j=0}^{3} e^{-i6\pi j/4}|j\rangle\right) \otimes |1\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1 + e^{-i4\pi j/4}\right)|j\rangle\right) \otimes |0\rangle + \left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4} + e^{-i6\pi j/4}\right)|j\rangle\right) \otimes |1\rangle\right) \\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle \otimes \left(|0\rangle + e^{-i4\pi j/4}|0\rangle + e^{-i2\pi j/4}|1\rangle + e^{-i6\pi j/4}|1\rangle\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + |2\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{0} \to \frac{4}{4} = 1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2} = 2$ which appears to be the correct underlying periodicity. Compare this result to the answer for Assignment 9.

$$
\begin{aligned}
(U_{QFT,4} \otimes I_2)|\psi_2\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle) \otimes |1\rangle + (U_{QFT,4}|1\rangle) \otimes |0\rangle + (U_{QFT,4}|2\rangle) \otimes |1\rangle + (U_{QFT,4}|3\rangle) \otimes |0\rangle) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right) \otimes |1\rangle + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right) \otimes |0\rangle\right. \\
&\quad \left. + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right) \otimes |1\rangle + \left(\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right) \otimes |0\rangle\right) \\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right) \otimes |1\rangle + \left(\sum_{j=0}^{3} e^{-i2\pi j/4}|j\rangle\right) \otimes |0\rangle\right. \\
&\quad \left. + \left(\sum_{j=0}^{3} e^{-i4\pi j/4}|j\rangle\right) \otimes |1\rangle + \left(\sum_{j=0}^{3} e^{-i6\pi j/4}|j\rangle\right) \otimes |0\rangle\right)
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1+e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|1\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|0\rangle\right)\\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|0\rangle+e^{-i4\pi j/4}|1\rangle+e^{-i2\pi j/4}|1\rangle+e^{-i6\pi j/4}|0\rangle\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|1\rangle+|0\rangle)+|2\rangle\otimes\frac{1}{\sqrt{2}}(|1\rangle-|0\rangle)\right)\\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)-|2\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right).
\end{aligned}$$

Thus the periodicity is deduced from $\frac{4}{0}\to\frac{4}{4}=1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2}=2$ which appears to be the correct underlying periodicity. Compare this result to the answer for Assignment 9.

2. For $U_{IA,1}$ we find

$$U_{IA,1}=U_H\begin{pmatrix}1&0\\0&-1\end{pmatrix}U_H=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$U_{IA,1}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}b\\a\end{pmatrix}.$$

Noticing that the average of a and b is $(a+b)/2$ we find the inverse about the average for

$$a=(a+b)/2+(a-b)/2$$

is

$$(a+b)/2-(a-b)/2=b.$$

Similiarly the inverse about the average for

$$b=(a+b)/2+(b-a)/2$$

is

$$(a+b)/2-(b-a)/2=a.$$

For $U_{IA,2}$ we find

$$\begin{aligned}
U_{IA,2} &= (U_H\otimes U_H)\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}(U_H\otimes U_H)\\
&= \frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}-1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{pmatrix}
\end{aligned}$$

$$U_{IA,2}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\frac{1}{2}\begin{pmatrix}b+c+d-a\\a+c+d-b\\a+b+d-c\\a+b+c-d\end{pmatrix}.$$

The average of a, b, c and d is $(a+b+c+d)/4$. Thus we have

$$a=\frac{a+b+c+d}{4}+\frac{3a-b-c-d}{4},\quad b=\frac{a+b+c+d}{4}+\frac{3b-a-c-d}{4},$$

$$c=\frac{a+b+c+d}{4}+\frac{3c-a-b-d}{4},\quad d=\frac{a+b+c+d}{4}+\frac{3d-a-b-c}{4}.$$

The inverse about the average in each case is

$$\frac{a+b+c+d}{4} - \frac{3a-b-c-d}{4} = \frac{b+c+d-a}{2}, \qquad \frac{a+b+c+d}{4} - \frac{3b-a-c-d}{4} = \frac{a+c+d-b}{2},$$

$$\frac{a+b+c+d}{4} - \frac{3c-a-b-d}{4} = \frac{a+b+d-c}{2}, \qquad \frac{a+b+c+d}{4} - \frac{3d-a-b-c}{4} = \frac{a+b+c-d}{2}.$$

3. We have

$$\langle 0|0\rangle = \Big(\langle 0|a\rangle\langle a| + \langle 0|b\rangle\langle b|\Big)\Big(\langle a|0\rangle|a\rangle + \langle b|0\rangle|b\rangle\Big) = |\langle a|0\rangle|^2 + |\langle b|0\rangle|^2 = 1$$

$$\langle 1|1\rangle = \Big(\langle 1|a\rangle\langle a| + \langle 1|b\rangle\langle b|\Big)\Big(\langle a|1\rangle|a\rangle + \langle b|1\rangle|b\rangle\Big) = |\langle a|1\rangle|^2 + |\langle b|1\rangle|^2 = 1$$

and

$$\langle 0|1\rangle = \Big(\langle 0|a\rangle\langle a| + \langle 0|b\rangle\langle b|\Big)\Big(\langle a|1\rangle|a\rangle + \langle b|1\rangle|b\rangle\Big) = \langle 0|a\rangle\langle a|1\rangle + \langle 0|b\rangle\langle b|1\rangle = 0.$$

For $|\psi\rangle$ we find

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle) \\
&= \frac{1}{\sqrt{2}}\Bigg(\Big(\langle a|0\rangle|a\rangle + \langle b|0\rangle|b\rangle\Big)\otimes\Big(\langle a|1\rangle|a\rangle + \langle b|1\rangle|b\rangle\Big) - \Big(\langle a|1\rangle|a\rangle + \langle b|1\rangle|b\rangle\Big)\otimes\Big(\langle a|0\rangle|a\rangle + \langle b|0\rangle|b\rangle\Big)\Bigg) \\
&= \frac{1}{\sqrt{2}}\big(\langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle\big)\Big(|a\rangle\otimes|b\rangle - |b\rangle\otimes|a\rangle\Big).
\end{aligned}$$

Notice that

$$\begin{aligned}
|\langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle|^2 &= \Big(\langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle\Big)\Big(\langle 0|a\rangle\langle 1|b\rangle - \langle 1|a\rangle\langle 0|b\rangle\Big) \\
&= |\langle a|0\rangle|^2|\langle b|1\rangle|^2 - \langle a|0\rangle\langle b|1\rangle\langle 1|a\rangle\langle 0|b\rangle - \langle a|1\rangle\langle b|0\rangle\langle 0|a\rangle\langle 1|b\rangle + |\langle a|1\rangle|^2|\langle b|0\rangle|^2 \\
&= \big(1 - |\langle b|0\rangle|^2|\big)\langle b|1\rangle|^2 + \langle b|0\rangle\langle b|1\rangle\langle 1|b\rangle\langle 0|b\rangle \\
&\quad + \langle b|1\rangle\langle b|0\rangle\langle 0|b\rangle\langle 1|b\rangle + \big(1 - |\langle b|1\rangle|^2\big)|\langle b|0\rangle|^2 \\
&= |\langle b|0\rangle|^2 + |\langle b|1\rangle|^2 = 1
\end{aligned}$$

where we used the results for $\langle 0|0\rangle$, $\langle 1|1\rangle$ and $\langle 0|1\rangle$. We define for convenience

$$c := \langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle$$

where $|c| = 1$.

Refer to Assignment 5 for a detailed example of measurement. Here we list the results of measurement only. The measurent outcome 0 with respect to the observable $I_2\otimes|1\rangle\langle 1|$ (respectively $I_2\otimes|b\rangle\langle b|$) corresponds to (a projection onto) the state $|0\rangle$ (respectively $|a\rangle$) for the second qubit. Similarly the measurent outcome 1 with respect to the observable $I_2\otimes|1\rangle\langle 1|$ (respectively $I_2\otimes|b\rangle\langle b|$) corresponds to (a projection onto) the state $|1\rangle$ (respectively $|b\rangle$) for the second qubit.

The following table lists all possibilities with the associated probabilities, and the total probability of obtaining different outcomes for the measurement. If

$$|0\rangle = \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|b\rangle, \qquad |1\rangle = \frac{1}{\sqrt{2}}|a\rangle - \frac{1}{\sqrt{2}}|b\rangle$$

or equivalently

$$|a\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \qquad |b\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

then we find the total probability

$$\frac{1}{2} + \frac{|\langle 0|a\rangle|^2}{4} + \frac{|\langle 1|b\rangle|^2}{4} = \frac{3}{4}.$$

Observ.	Prob.	Outcome	Prob.	Projection	Observ.	Prob.	Outcome	Prob.	Total prob.
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	0	0	
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	**0**	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	**1**	1	$\frac{1}{8}$
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	**1**	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	**0**	1	$\frac{1}{8}$
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	1	0	
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	0	$\|\langle a\|1\rangle\|^2$	
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	**0**	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	**1**	$\|\langle b\|1\rangle\|^2$	$\frac{\|\langle b\|1\rangle\|^2}{8}$
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	**1**	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	**0**	$\|\langle a\|0\rangle\|^2$	$\frac{\|\langle a\|0\rangle\|^2}{8}$
$I_2 \otimes \|1\rangle\langle 1\|$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	1	$\|\langle b\|0\rangle\|^2$	
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	0	$\|\langle 0\|b\rangle\|^2$	
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	**0**	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	**1**	$\|\langle 1\|b\rangle\|^2$	$\frac{\|\langle 1\|b\rangle\|^2}{8}$
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	**1**	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	**0**	$\|\langle 0\|a\rangle\|^2$	$\frac{\|\langle 0\|a\rangle\|^2}{8}$
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	1	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	$\|1\rangle\langle 1\| \otimes I_2$	$\frac{1}{2}$	1	$\|\langle 1\|a\rangle\|^2$	
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	0	0	
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	**0**	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	**1**	1	$\frac{1}{8}$
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	**1**	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	**0**	1	$\frac{1}{8}$
$I_2 \otimes \|b\rangle\langle b\|$	$\frac{1}{2}$	1	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	$\|b\rangle\langle b\| \otimes I_2$	$\frac{1}{2}$	1	0	
Total:									$\frac{1}{2} + \frac{\|\langle 0\|a\rangle\|^2}{4} + \frac{\|\langle 1\|b\rangle\|^2}{4}$