# University of Johannesburg

## Applied Mathematics 3B

**Semester test 2** **Solutions**

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1. (a) Since $\rho^* = \rho$, $\rho$ is Hermitian. Obviously $\text{tr}\rho = 1$. It remains to determine whether $\rho$ is positive semi-definite. The eigenvalues of $\rho$ follow from

$$\begin{aligned}
\det(\lambda I_4 - \rho) &= \det\begin{pmatrix} \lambda - \frac{1}{4} & 0 & 0 & -\frac{i}{4} \\ 0 & \lambda - \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \lambda - \frac{1}{4} & 0 \\ \frac{i}{4} & 0 & 0 & \lambda - \frac{1}{4} \end{pmatrix} \\
&= \left(\lambda - \frac{1}{4}\right) \det\begin{pmatrix} \lambda - \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \lambda - \frac{1}{4} & 0 \\ 0 & 0 & \lambda - \frac{1}{4} \end{pmatrix} + \frac{i}{4}\det\begin{pmatrix} 0 & \lambda - \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \lambda - \frac{1}{4} \\ \frac{i}{4} & 0 & 0 \end{pmatrix} \\
&= \left(\lambda - \frac{1}{4}\right)\left[\left(\lambda - \frac{1}{4}\right)^3 - \frac{1}{16}\left(\lambda - \frac{1}{4}\right)\right] + \frac{i}{4}\left[\frac{i}{4}\left(\lambda - \frac{1}{4}\right)^2 - \frac{i}{64}\right] \\
&= \left(\lambda - \frac{1}{4}\right)^2\left[\left(\lambda - \frac{1}{4}\right)^2 - \frac{1}{16}\right] - \frac{1}{16}\left[\left(\lambda - \frac{1}{4}\right)^2 - \frac{1}{16}\right] \\
&= \left[\left(\lambda - \frac{1}{4}\right)^2 - \frac{1}{16}\right]^2 = 0
\end{aligned}$$

It follows that

$$\lambda = \pm\frac{1}{4} + \frac{1}{4} \quad \Rightarrow \quad \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = \frac{1}{2}, \lambda_4 = \frac{1}{2}.$$

All the eigenvalues are non-negative, so $\rho$ is a density matrix.

(b) We have

$$\rho_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously $\rho_1$ is Hermitian, has unit trace and the eigenvalues are $\frac{1}{2}$ and $\frac{1}{2}$, so $\rho_1$ is a density matrix.

(c) Since $\rho_1$ and $\rho_2$ are density matrices we have $\text{tr}\rho_1 = \text{tr}\rho_2 = 1$. Thus

$$\text{tr}(\rho_1 + \rho_2) = \text{tr}\rho_1 + \text{tr}\rho_2 = 2.$$

Thus $\rho_1 + \rho_2$ is not a density operator.

(d) Consider $n = 2$ and $\rho_1 = \rho_2 = \frac{1}{2}I_2$. The $\text{tr}(\rho_1\rho_2) = \frac{1}{4}\text{tr}I_2 = \frac{1}{2}$ i.e. we have a counter example. Thus $\rho_1\rho_2$ is not a density matrix *in general*.

(Note that there are special cases for, for example $\rho_1 = \rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ provides the density matrix $\rho_1\rho_2 = \rho_1$.)

2. We have

$$U_{QFT,3}|\psi_1\rangle = \frac{1}{3}\sum_{j=0}^{2}\sum_{k=0}^{2} e^{-i2\pi jk/3}|j\rangle\langle k|0\rangle + \frac{1}{3}\sum_{j=0}^{2}\sum_{k=0}^{2} e^{-i2\pi jk/3}|j\rangle\langle k|1\rangle + \frac{1}{3}\sum_{j=0}^{2}\sum_{k=0}^{2} e^{-i2\pi jk/3}|j\rangle\langle k|2\rangle$$

$$
\begin{aligned}
&= \frac{1}{3}\sum_{j=0}^{2} e^{-i2\pi j\cdot 0/3}|j\rangle + \frac{1}{3}\sum_{j=0}^{2} e^{-i2\pi j\cdot 1/3}|j\rangle + \frac{1}{3}\sum_{j=0}^{2} e^{-i2\pi j\cdot 2/3}|j\rangle \\
&= \frac{1}{3}\sum_{j=0}^{2}\left(e^{-i2\pi j\cdot 0/3} + e^{-i2\pi j\cdot 1/3} + e^{-i2\pi j\cdot 2/3}\right)|j\rangle \\
&= \frac{1}{3}(1+1+1)|0\rangle + \frac{1}{3}\left(1 - \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)|1\rangle + \frac{1}{3}\left(1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)|1\rangle \\
&= |0\rangle.
\end{aligned}
$$

We identify the period from the non-zero coefficients, namely the coefficient of $|0\rangle$. Replacing 0 with 3 for the division we find that the period is $\frac{3}{3} = 1$.

We have

$$
\begin{aligned}
U_{QFT,3}|\psi_2\rangle &= \frac{1}{3}\sum_{j=0}^{2}\sum_{k=0}^{2} e^{-i2\pi jk/3}|j\rangle\langle k|0\rangle - \frac{1}{3}\sum_{j=0}^{2}\sum_{k=0}^{2} e^{-i2\pi jk/3}|j\rangle\langle k|1\rangle + \frac{1}{3}\sum_{j=0}^{2}\sum_{k=0}^{2} e^{-i2\pi jk/3}|j\rangle\langle k|2\rangle \\
&= \frac{1}{3}\sum_{j=0}^{2} e^{-i2\pi j\cdot 0/3}|j\rangle - \frac{1}{3}\sum_{j=0}^{2} e^{-i2\pi j\cdot 1/3}|j\rangle + \frac{1}{3}\sum_{j=0}^{2} e^{-i2\pi j\cdot 2/3}|j\rangle \\
&= \frac{1}{3}\sum_{j=0}^{2}\left(e^{-i2\pi j\cdot 0/3} - e^{-i2\pi j\cdot 1/3} + e^{-i2\pi j\cdot 2/3}\right)|j\rangle \\
&= \frac{1}{3}(1-1+1)|0\rangle + \frac{1}{3}\left(1 + \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)|1\rangle + \frac{1}{3}\left(1 + \frac{1}{2} - i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)|1\rangle \\
&= \frac{1}{3}|0\rangle + \frac{1}{3}(1+i\sqrt{3})|1\rangle + \frac{1}{3}(1-i\sqrt{3})|2\rangle.
\end{aligned}
$$

We identify the period from the non-zero coefficients, however all coefficients are non-zero. Thus all periods are present (i.e. 1, 3, and $\frac{3}{2}$ (which does not make sense)). We assume that the coefficients sequence are truncated from an infinite sequence. Furthermore 3 is prime, so that the only periods that make sense for the truncated sequence are 1 and 3. We cannot determine the period of this sequence (it is not periodic over 3 values).

3. Since

$$|s_j\rangle_n = \sum_{k=0}^{n-1} \delta_{s_j,k}|k\rangle_n$$

we find

$$
\begin{aligned}
|\mathbf{s}\rangle &= \sum_{j=0}^{m-1} |j\rangle_m \otimes |s_j\rangle_n = \sum_{j=0}^{m-1} |j\rangle_m \otimes \sum_{k=0}^{n-1} \delta_{s_j,k}|k\rangle_n \\
&= \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} \delta_{s_j,k}|j\rangle_m \otimes |k\rangle_n = \sum_{k=0}^{n-1}\left(\sum_{j=0}^{m-1} \delta_{s_j,k}|j\rangle_m\right) \otimes |k\rangle_n.
\end{aligned}
$$

Let $f(x)$ be a function and $x_0, x_1, \ldots$ be a sequence. Let $p$ be the period of $x_0, x_1, \ldots$, i.e. $x_{j+p} = x_j$. then $f(x_0), f(x_1), \ldots$ is also $p$-periodic i.e. $f(x_{j+p}) = f(x_j)$. This might not be the minimal period.

It follows that

$$\mathbf{s}_k = \delta_{s_0,k}, \delta_{s_1,k}, \ldots, \delta_{s_{m-1},k}$$

is $p$-periodic if $\mathbf{s}$ is $p$-periodic, since $\delta_{s_j,k}$ is the Kronecker delta function.