University of Johannesburg

Applied Mathematics 3B

Semester test 1 **Solutions**

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1. We have

$$
\begin{aligned}
&\left[\frac{1}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\right]^2 \\
&= \frac{1}{(4-2\sqrt{2})^2}\begin{pmatrix} (3-2\sqrt{2})^2+(1-\sqrt{2})^2 & (3-2\sqrt{2})(1-\sqrt{2})+(1-\sqrt{2}) \\ (3-2\sqrt{2})(1-\sqrt{2})+(1-\sqrt{2}) & (1-\sqrt{2})^2+1 \end{pmatrix} \\
&= \frac{1}{(4-2\sqrt{2})^2}\begin{pmatrix} 20-14\sqrt{2} & (4-2\sqrt{2})(1-\sqrt{2}) \\ (4-2\sqrt{2})(1-\sqrt{2}) & 4-2\sqrt{2} \end{pmatrix} \\
&= \frac{1}{(4-2\sqrt{2})^2}\begin{pmatrix} (4-2\sqrt{2})(3-2\sqrt{2}) & (4-2\sqrt{2})(1-\sqrt{2}) \\ (4-2\sqrt{2})(1-\sqrt{2}) & 4-2\sqrt{2} \end{pmatrix} \\
&= \frac{1}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}
\end{aligned}
$$

where we used $(4-2\sqrt{2})(3-2\sqrt{2}) = 20-14\sqrt{2}$. The solution to the differential equation

$$\frac{d\psi}{dt} = \frac{i}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\psi$$

is

$$
\begin{aligned}
\psi(t) &= \exp\left(\frac{it}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\right)\psi(t=0) \\
&= \left(\sum_{j=0}^{\infty}\frac{(it)^j}{j!}\left[\frac{1}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\right]^j\right)\psi(t=0) \\
&= \left(I_2 + \left(\sum_{j=1}^{\infty}\frac{(it)^j}{j!}\right)\left[\frac{1}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\right]\right)\psi(t=0) \\
&= \left(I_2 + (e^{it}-1)\left[\frac{1}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\right]\right)\psi(t=0).
\end{aligned}
$$

At $t=\pi$ we find

$$
\begin{aligned}
\psi(t=\pi) &= \left(I_2 - 2\left[\frac{1}{4-2\sqrt{2}}\begin{pmatrix} 3-2\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}\right]\right)\psi(t=0) \\
&= \frac{1}{2-\sqrt{2}}\begin{pmatrix} \sqrt{2}-1 & \sqrt{2}-1 \\ \sqrt{2}-1 & 1-\sqrt{2} \end{pmatrix}\psi(t=0) \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\psi(t=0)
\end{aligned}
$$

i.e. we have performed the Hadamard transform. Thus we find

$$
\begin{aligned}
\psi(t=0) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \psi(t=\pi) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\psi(t=0) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \psi(t=\pi) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{aligned}
$$

2. Since

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\ 0 & \sqrt{2} & 0\\ 1 & 0 & -1\end{pmatrix}^* = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\ 0 & \sqrt{2} & 0\\ 1 & 0 & -1\end{pmatrix}$$

the matrix describes an observable. It is straightforward to determine that the eigenvalues of

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\\ 0 & \sqrt{2} & 0\\ 1 & 0 & -1\end{pmatrix}$$

are 1, 1 and -1 with corresponding orthonormal eigenvectors

$$\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix}1+\sqrt{2}\\0\\1\end{pmatrix}, \qquad \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}1-\sqrt{2}\\0\\1\end{pmatrix}.$$

Thus we have the measurement outcome 1 with probability

$$\left|\begin{pmatrix}0\\1\\0\end{pmatrix}^* \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right|^2 + \left|\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix}1+\sqrt{2}\\0\\1\end{pmatrix}^* \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right|^2 = \frac{1}{3}+\frac{2+\sqrt{2}}{6} = \frac{2}{3}+\frac{\sqrt{2}}{6}$$

and the outcome -1 with probability

$$\left|\frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix}1-\sqrt{2}\\0\\1\end{pmatrix}^* \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right|^2 = \frac{1}{3}+\frac{2-\sqrt{2}}{6} = \frac{2}{3}-\frac{\sqrt{2}}{6}.$$

The corresponding projections are

$$\Pi_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}^* + \frac{1}{4+2\sqrt{2}}\begin{pmatrix}1+\sqrt{2}\\0\\1\end{pmatrix}\begin{pmatrix}1+\sqrt{2}\\0\\1\end{pmatrix}^* = \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{4+2\sqrt{2}}\begin{pmatrix}3+2\sqrt{2} & 0 & 1+\sqrt{2}\\ 0 & 0 & 0\\ 1+\sqrt{2} & 0 & 1\end{pmatrix}$$

and

$$\Pi_{-1} = \frac{1}{4-2\sqrt{2}}\begin{pmatrix}1-\sqrt{2}\\0\\1\end{pmatrix}\begin{pmatrix}1-\sqrt{2}\\0\\1\end{pmatrix}^* = \frac{1}{4-2\sqrt{2}}\begin{pmatrix}3-2\sqrt{2} & 0 & 1-\sqrt{2}\\ 0 & 0 & 0\\ 1-\sqrt{2} & 0 & 1\end{pmatrix}.$$

If the measurement outcome was 1 then the state after measurement is

$$\frac{\Pi_1 \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}}{\left\|\Pi_1 \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|} = \sqrt{\frac{6+4\sqrt{2}}{16+11\sqrt{2}}}\begin{pmatrix}\frac{4+3\sqrt{2}}{4+2\sqrt{2}}\\ 1\\ \frac{1}{2}\end{pmatrix}.$$

If the measurement outcome was -1 then the state after measurement is

$$\frac{\Pi_{-1} \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}}{\left\|\Pi_{-1} \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|} = \sqrt{\frac{6-4\sqrt{2}}{10-7\sqrt{2}}}\begin{pmatrix}\frac{4-3\sqrt{2}}{4-2\sqrt{2}}\\ 0\\ \frac{1}{2}\end{pmatrix}.$$

3. We have

$$\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}^T \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 2 & 2\\ 2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix}$$

With eigenvalues 6, 0 and 0 and corresponding orthonormal eigenvectors

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \qquad \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}.$$

The singular values are $\sqrt{6}$, 0 and 0. Thus we have

$$V = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

The first column of U follows from the first columns of V

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and the second is chosen to be orthonormal to the first. It follows that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}^T.$$

The decomposition is obviously not unique, for example we can exchange the last two columns of V,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix}^T$$

or make a different choice for the second column of U

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}^T$$

or both

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix}^T$$

amongst other possibilities.

4. Let

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad C := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \qquad D := \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

where $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{C}$ for $i, j \in \{1, 2\}$. We find

$$A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

and

$$\frac{1}{2}\left(I_2 \otimes I_2 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Equating these two matrices yields $a_{11}b_{11} = 1$ and $a_{22}b_{22} = 1$ which imply $a_{11} \neq 0$ and $b_{11} \neq 0$. Consequently the equation $a_{11}b_{22} = 0$ cannot be satisfied. Thus no such A and B can be found.

From

$$C \otimes D = \begin{pmatrix} c_{11}d_{11} & c_{11}d_{12} & c_{12}d_{11} & c_{12}d_{12} \\ c_{11}d_{21} & c_{11}d_{22} & c_{12}d_{21} & c_{12}d_{22} \\ c_{21}d_{11} & c_{21}d_{12} & c_{22}d_{11} & c_{22}d_{12} \\ c_{21}d_{21} & c_{21}d_{22} & c_{22}d_{21} & c_{22}d_{22} \end{pmatrix}$$

and

$$a\,I_2\otimes I_2+b\,\sigma_x\otimes\sigma_x+c\,\sigma_y\otimes\sigma_y+d\,\sigma_z\otimes\sigma_z=\begin{pmatrix} a+d & 0 & 0 & b-c\\ 0 & a-d & b+c & 0\\ 0 & b+c & a-d & 0\\ b-c & 0 & 0 & a+d\end{pmatrix},$$

where a_{11} and a are independent of each other, we find that

$$C\otimes D=a\,I_2\otimes I_2+b\,\sigma_x\otimes\sigma_x+c\,\sigma_y\otimes\sigma_y+d\,\sigma_z\otimes\sigma_z$$

leads to the equations

$$\begin{array}{cccc} c_{11}d_{11}=a+d & c_{11}d_{12}=0 & c_{12}d_{11}=0 & c_{12}d_{12}=b-c\\ c_{11}d_{21}=0 & c_{11}d_{22}=a-d & c_{12}d_{21}=b+c & c_{12}d_{22}=0\\ c_{21}d_{11}=0 & c_{21}d_{12}=b+c & c_{22}d_{11}=a-d & c_{22}d_{12}=0\\ c_{21}d_{21}=b-c & c_{21}d_{22}=0 & c_{22}d_{21}=0 & c_{22}d_{22}=a+d\end{array}$$

The existence of solutions depends on the choice of C and D. In the following we assume $C\neq 0_2$ and $D\neq 0_2$ since $C=0_2$ trivially provides the solution $a=b=c=d=0$ (similarly for $D=0_2$. This possibility will be incorporated in the final solutions.

A solution exists provided

$$\begin{pmatrix} c_{11} & 0\\ 0 & c_{22}\end{pmatrix}\otimes\begin{pmatrix} 0 & d_{12}\\ d_{21} & 0\end{pmatrix}=\begin{pmatrix} 0 & c_{12}\\ c_{21} & 0\end{pmatrix}\otimes\begin{pmatrix} d_{11} & 0\\ 0 & d_{22}\end{pmatrix}=0_4$$

where 0_4 is the 4×4 zero matrix.

In other words C must be diagonal or have zeroes on the diagonal (similarly for D). Suppose C is non-zero and diagonal, then the above equation implies D is diagonal, $b=c=0$ and

$$a=\frac{1}{2}c_{11}(d_{11}+d_{22})=\frac{1}{2}c_{22}(d_{11}+d_{22}),\quad d=\frac{1}{2}c_{11}(d_{11}-d_{22})=-\frac{1}{2}c_{22}(d_{11}-d_{22}).$$

For the above equalities to hold we must have $C=\alpha I_2$ and $D=\beta I_2$, or $C=\alpha\sigma_z$ and $D=\beta\sigma_z$ where $\alpha,\beta\in\mathbb{C}$ are arbitrary.

Similarly if C has zeroes on the diagonal, then D has zeroes on the diagonal, $a=d=0$ and

$$b=\frac{1}{2}c_{12}(d_{12}+d_{21})=\frac{1}{2}c_{21}(d_{12}+d_{21}),\quad c=\frac{1}{2}c_{12}(d_{21}-d_{12})=-\frac{1}{2}c_{21}(d_{21}-d_{12}).$$

For the above equalities to hold we must have $C=\alpha\sigma_x$ and $D=\beta\sigma_x$, or $C=\alpha\sigma_y$ and $D=\beta\sigma_y$ where $\alpha,\beta\in\mathbb{C}$ are arbitrary.

To summarize

$$\begin{array}{rcl} C=\alpha I_2,\ D=\beta I_2 & \Rightarrow & a=\alpha\beta\,, b=c=d=0\\ C=\alpha\sigma_x,\ D=\beta\sigma_x & \Rightarrow & b=\alpha\beta,\ a=c=d=0\\ C=\alpha\sigma_y,\ D=\beta\sigma_y & \Rightarrow & c=\alpha\beta,\ a=b=d=0\\ C=\alpha\sigma_z,\ D=\beta\sigma_z & \Rightarrow & d=\alpha\beta,\ a=b=c=0\end{array}$$

where $\alpha,\beta\in\mathbb{C}$ are arbitrary. For all other C and D no solution exists.