University of Johannesburg

Applied Mathematics 3B

November Exam **Solutions**

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1. (a) Since $\rho^* \neq \rho$, ρ is not Hermitian. Thus ρ is not a density matrix.

   (b) We have

$$\rho_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   Obviously ρ_2 is Hermitian, has unit trace and the eigenvalues are $\frac{1}{2}$ and $\frac{1}{2}$, so ρ_2 is a density matrix.

2. (a) We have

$$\begin{aligned} \|u\mathbf{x} - \mathbf{a} \otimes \mathbf{b}\|^2 &= (u\mathbf{x} - \mathbf{a} \otimes \mathbf{b})^T (u\mathbf{x} - \mathbf{a} \otimes \mathbf{b}) \\ &= |u|^2 \left(\mathbf{x} - \frac{1}{u}\mathbf{a} \otimes \mathbf{b}\right)^T \left(\mathbf{x} - \frac{1}{u}\mathbf{a} \otimes \mathbf{b}\right) \\ &= |u|^2 \left(\mathbf{x} - \left(\frac{1}{u}\mathbf{a}\right) \otimes \mathbf{b}\right)^T \left(\mathbf{x} - \left(\frac{1}{u}\mathbf{a}\right) \otimes \mathbf{b}\right) \\ &= |u|^2 \left\|\mathbf{x} - \left(\frac{1}{u}\mathbf{a}\right) \otimes \mathbf{b}\right\|^2. \end{aligned}$$

   Thus the minimum of $\|u\mathbf{x} - \mathbf{a} \otimes \mathbf{b}\|^2$ is $|u|^2$ times the minimum of $\|\mathbf{x} - \tilde{\mathbf{a}} \otimes \mathbf{b}\|^2$ where $\mathbf{a} = u\tilde{\mathbf{a}}$.

   (b) Since

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

   we consider

$$\begin{aligned} g(a_1, a_2, b_1, b_2) := \left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \mathbf{a} \otimes \mathbf{b} \right\|^2 &= \left(\frac{1}{\sqrt{2}} - a_1 b_1\right)^2 + (-a_1 b_2)^2 + (-a_2 b_1)^2 + \left(\frac{1}{\sqrt{2}} - a_2 b_2\right)^2 \\ &= \left(a_1 b_1 - \frac{1}{\sqrt{2}}\right)^2 + (a_1 b_2)^2 + (a_2 b_1)^2 + \left(a_2 b_2 - \frac{1}{\sqrt{2}}\right)^2. \end{aligned}$$

   To minimize we must solve

$$\begin{aligned} \frac{\partial g}{\partial a_1} &= 2\left(a_1 b_1 - \frac{1}{\sqrt{2}}\right) b_1 + 2a_1 b_2^2 = 0 \\ \frac{\partial g}{\partial a_2} &= 2a_2 b_1^2 + 2\left(a_2 b_2 - \frac{1}{\sqrt{2}}\right) b_2 = 0 \\ \frac{\partial g}{\partial b_1} &= 2\left(a_1 b_1 - \frac{1}{\sqrt{2}}\right) a_1 + 2a_2^2 b_1 = 0 \\ \frac{\partial g}{\partial b_2} &= 2a_1^2 b_2 + 2\left(a_2 b_2 - \frac{1}{\sqrt{2}}\right) a_2 = 0 \end{aligned}$$

   These equations can be rewritten as

$$b_1 = \sqrt{2} a_1 (b_1^2 + b_2^2), \qquad b_2 = \sqrt{2} a_2 (b_1^2 + b_2^2), \qquad a_1 = \sqrt{2} b_1 (a_1^2 + a_2^2), \qquad a_2 = \sqrt{2} b_2 (a_1^2 + a_2^2)$$

If $b_1^2+b_2^2=0$ then $a_1=a_2=b_1=b_2=0$ and $g(0,0,0,0)=2$. Now consider $b_1^2+b_2^2\neq 0$. It follows that

$$a_1=\frac{1}{\sqrt{2}}\frac{b_1}{b_1^2+b_2^2},\qquad a_2=\frac{1}{\sqrt{2}}\frac{b_2}{b_1^2+b_2^2}.$$

For these values we find

$$g\left(\frac{1}{\sqrt{2}}\frac{b_1}{b_1^2+b_2^2},\frac{1}{\sqrt{2}}\frac{b_2}{b_1^2+b_2^2},b_1,b_2\right)=\frac{1}{2}.$$

To determine whether this is the global minimum we first note that

$$g(a_1,a_2,b_1,b_2)=(\mathbf{a}^T\mathbf{a})(\mathbf{b}^T\mathbf{b})-\sqrt{2}\mathbf{a}^T\mathbf{b}+1.$$

The Cauchy-Schwarz inequality provides

$$(\mathbf{a}^T\mathbf{a})(\mathbf{b}^T\mathbf{b})\geq(\mathbf{a}^T\mathbf{b})^2$$

so that

$$g(a_1,a_2,b_1,b_2)\geq(\mathbf{a}^T\mathbf{b})^2-\sqrt{2}\mathbf{a}^T\mathbf{b}+1=\left(\mathbf{a}^T\mathbf{b}-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}\geq\frac{1}{2}.$$

Thus the global minimum is found at

$$\mathbf{a}=\frac{1}{b_1^2+b_2^2}\mathbf{b},\qquad \mathbf{b}\in\mathbb{R}^2/\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}.$$

This result is consistent with (a) (we could also have applied (a) to obtain this result, although there is no substantial advantage in doing this).

(c) We could proceed as above.

$$\begin{aligned}h(a_1,a_2,b_1,b_2):=\left\|\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}-\mathbf{a}\otimes\mathbf{b}\right\|^2 &= \left(\frac{1}{\sqrt{2}}-a_1b_1\right)^2+(-a_1b_2)^2+(-a_2b_1)^2+\left(-\frac{1}{\sqrt{2}}-a_2b_2\right)^2\\ &= \left(a_1b_1-\frac{1}{\sqrt{2}}\right)^2+(a_1b_2)^2+(a_2b_1)^2+\left(a_2b_2+\frac{1}{\sqrt{2}}\right)^2.\end{aligned}$$

However, noting that $h(a_1,a_2,b_1,b_2)=g(a_1,a_2,b_1,-b_2)$ yields the minimum

$$h(a_1,a_2,b_1,b_2)=\frac{1}{2}$$

at

$$\mathbf{a}=\frac{1}{b_1^2+b_2^2}\begin{pmatrix}b_1\\-b_2\end{pmatrix},\qquad \mathbf{b}\in\mathbb{R}^2/\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$$

directly.

3. We have

$$\begin{aligned}(U_{QFT,4}\otimes I_2)|\psi_1\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle)\otimes|0\rangle+(U_{QFT,4}|1\rangle)\otimes|1\rangle+(U_{QFT,4}|2\rangle)\otimes|0\rangle+(U_{QFT,4}|3\rangle)\otimes|1\rangle)\\ &= \frac{1}{4}\Bigg(\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right)\otimes|1\rangle\\ &\quad+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right)\otimes|1\rangle\Bigg)\\ &= \frac{1}{4}\Bigg(\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|1\rangle\\ &\quad+\left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|1\rangle\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1+e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|0\rangle+e^{-i4\pi j/4}|0\rangle+e^{-i2\pi j/4}|1\rangle+e^{-i6\pi j/4}|1\rangle\right)\right)\\
&= \frac{1}{\sqrt{2}}\left(|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)+|2\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{0}\to\frac{4}{4}=1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), and $\frac{4}{2}=2$ which appears to be the correct underlying periodicity.

$$
\begin{aligned}
(U_{QFT,4}\otimes I_2)|\psi_2\rangle &= \frac{1}{2}((U_{QFT,4}|0\rangle)\otimes|0\rangle+(U_{QFT,4}|1\rangle)\otimes|1\rangle+(U_{QFT,4}|2\rangle)\otimes|1\rangle+(U_{QFT,4}|3\rangle)\otimes|0\rangle)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|0\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|1\rangle\right)\otimes|1\rangle\right.\\
&\qquad\left.+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|2\rangle\right)\otimes|1\rangle+\left(\sum_{j=0}^{3}\sum_{k=0}^{3}e^{-i2\pi jk/4}|j\rangle\langle k|3\rangle\right)\otimes|0\rangle\right)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}e^{-i2\pi j/4}|j\rangle\right)\otimes|1\rangle\right.\\
&\qquad\left.+\left(\sum_{j=0}^{3}e^{-i4\pi j/4}|j\rangle\right)\otimes|1\rangle+\left(\sum_{j=0}^{3}e^{-i6\pi j/4}|j\rangle\right)\otimes|0\rangle\right)\\
&= \frac{1}{4}\left(\left(\sum_{j=0}^{3}\left(1+e^{-i6\pi j/4}\right)|j\rangle\right)\otimes|0\rangle+\left(\sum_{j=0}^{3}\left(e^{-i2\pi j/4}+e^{-i4\pi j/4}\right)|j\rangle\right)\otimes|1\rangle\right)\\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left(|0\rangle+e^{-i4\pi j/4}|1\rangle+e^{-i2\pi j/4}|1\rangle+e^{-i6\pi j/4}|0\rangle\right)\right)\\
&= \frac{1}{4}\left(\sum_{j=0}^{3}|j\rangle\otimes\left((1+i^j)|0\rangle+((-1)^j+(-i)^j)|1\rangle\right)\right)\\
&= \frac{1}{4}\left(|0\rangle\otimes(2|1\rangle+2|0\rangle)+|1\rangle\otimes((1+i)|1\rangle-(1+i)|0\rangle)+|3\rangle\otimes((1-i)|1\rangle-(1-i)|0\rangle)\right).
\end{aligned}
$$

Thus the periodicity is deduced from $\frac{4}{0}\to\frac{4}{4}=1$ (we know this is not the case, but it results from truncation of the sequence and because the sequence is not composed of appropriate exponentials), $\frac{4}{1}=4$, and $\frac{4}{3}$ (does not make sense). This result is due to uncertainty about the truncated sequence, i.e. we could have

$$0110011001100110\ldots$$

or

$$0110110110110110\ldots$$

The first sequence has period 4 while the second has period 3.

4. We have

$$\langle 0|0\rangle=\left(\langle 0|a\rangle\langle a|+\langle 0|b\rangle\langle b|\right)\left(\langle a|0\rangle|a\rangle+\langle b|0\rangle|b\rangle\right)=|\langle a|0\rangle|^2+|\langle b|0\rangle|^2=1$$

$$\langle 1|1\rangle=\left(\langle 1|a\rangle\langle a|+\langle 1|b\rangle\langle b|\right)\left(\langle a|1\rangle|a\rangle+\langle b|1\rangle|b\rangle\right)=|\langle a|1\rangle|^2+|\langle b|1\rangle|^2=1$$

and

$$\langle 0|1\rangle=\left(\langle 0|a\rangle\langle a|+\langle 0|b\rangle\langle b|\right)\left(\langle a|1\rangle|a\rangle+\langle b|1\rangle|b\rangle\right)=\langle 0|a\rangle\langle a|1\rangle+\langle 0|b\rangle\langle b|1\rangle=0.$$

For $|\psi\rangle$ we find

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \\
&= \frac{1}{\sqrt{2}}\Big(\Big(\langle a|0\rangle|a\rangle + \langle b|0\rangle|b\rangle\Big) \otimes \Big(\langle a|1\rangle|a\rangle + \langle b|1\rangle|b\rangle\Big) - \Big(\langle a|1\rangle|a\rangle + \langle b|1\rangle|b\rangle\Big) \otimes \Big(\langle a|0\rangle|a\rangle + \langle b|0\rangle|b\rangle\Big)\Big) \\
&= \frac{1}{\sqrt{2}}\big(\langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle\big)\Big(|a\rangle \otimes |b\rangle - |b\rangle \otimes |a\rangle\Big).
\end{aligned}$$

Notice that

$$\begin{aligned}
|\langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle|^2 &= \Big(\langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle\Big)\Big(\langle 0|a\rangle\langle 1|b\rangle - \langle 1|a\rangle\langle 0|b\rangle\Big) \\
&= |\langle a|0\rangle|^2|\langle b|1\rangle|^2 - \langle a|0\rangle\langle b|1\rangle\langle 1|a\rangle\langle 0|b\rangle - \langle a|1\rangle\langle b|0\rangle\langle 0|a\rangle\langle 1|b\rangle + |\langle a|1\rangle|^2|\langle b|0\rangle|^2 \\
&= \big(1 - |\langle b|0\rangle|^2|\big)\langle b|1\rangle|^2 + \langle b|0\rangle\langle b|1\rangle\langle 1|b\rangle\langle 0|b\rangle \\
&\quad + \langle b|1\rangle\langle b|0\rangle\langle 0|b\rangle\langle 1|b\rangle + \big(1 - |\langle b|1\rangle|^2\big)|\langle b|0\rangle|^2 \\
&= |\langle b|0\rangle|^2 + |\langle b|1\rangle|^2 = 1
\end{aligned}$$

where we used the results for $\langle 0|0\rangle$, $\langle 1|1\rangle$ and $\langle 0|1\rangle$. We define for convenience

$$c := \langle a|0\rangle\langle b|1\rangle - \langle a|1\rangle\langle b|0\rangle$$

where $|c| = 1$. Similarly we define $\alpha_0 := \langle a|0\rangle/|\langle a|0\rangle|$ $\alpha_1 := \langle a|1\rangle/|\langle a|1\rangle|$ $\beta_0 := \langle b|0\rangle/|\langle b|0\rangle|$ $\beta_1 := \langle b|1\rangle/|\langle b|1\rangle|$. We can ignore division by zero, since these occur with zero probability.

Refer to Assignment 5 for a detailed example of measurement. Here we list the results of measurement only. The measurent outcome 0 with respect to the observable $I_2 \otimes |1\rangle\langle 1|$ (respectively $I_2 \otimes |b\rangle\langle b|$) corresponds to (a projection onto) the state $|0\rangle$ (respectively $|a\rangle$) for the second qubit. Similarly the measurent outcome 1 with respect to the observable $I_2 \otimes |1\rangle\langle 1|$ (respectively $I_2 \otimes |b\rangle\langle b|$) corresponds to (a projection onto) the state $|1\rangle$ (respectively $|b\rangle$) for the second qubit.

The tables are repetetive, but are all included for completeness (it would have been sufficient to consider only half of these).

Outcome	Prob.	Projection	Outcome	Prob.	Projection	Outcome	Prob.	Total prob.
	M_1			M_3			M_3	
0	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	0	0				
0	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	1	1	$\|0\rangle \otimes \|1\rangle$	**1**	1	$\frac{1}{2}$
1	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	0	1	$-\|1\rangle \otimes \|0\rangle$	**0**	1	$\frac{1}{2}$
1	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	1	0				
	M_1			M_3			M_4	
0	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	0	0				
0	$\frac{1}{2}$	$\|0\rangle \otimes \|1\rangle$	1	1	$\|0\rangle \otimes \|1\rangle$	**1**	$\|\langle b\|1\rangle\|^2$	$\frac{\|\langle b\|1\rangle\|^2}{2}$
1	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	0	1	$-\|1\rangle \otimes \|0\rangle$	**0**	$\|\langle a\|0\rangle\|^2$	$\frac{\|\langle a\|0\rangle\|^2}{2}$
1	$\frac{1}{2}$	$-\|1\rangle \otimes \|0\rangle$	1	0				
	M_2			M_4			M_4	
0	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	0	0				
0	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	1	1	$c\|a\rangle \otimes \|b\rangle$	**1**	1	$\frac{1}{2}$
1	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	0	1	$-c\|b\rangle \otimes \|a\rangle$	**0**	1	$\frac{1}{2}$
1	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	1	0				
	M_2			M_4			M_3	
0	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	0	0				
0	$\frac{1}{2}$	$c\|a\rangle \otimes \|b\rangle$	1	1	$c\|a\rangle \otimes \|b\rangle$	**1**	$\|\langle b\|1\rangle\|^2$	$\frac{\|\langle b\|1\rangle\|^2}{2}$
1	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	0	1	$-c\|b\rangle \otimes \|a\rangle$	**0**	$\|\langle a\|0\rangle\|^2$	$\frac{\|\langle a\|0\rangle\|^2}{2}$
1	$\frac{1}{2}$	$-c\|b\rangle \otimes \|a\rangle$	1	0				

Outcome	Prob.	Projection	Outcome	Prob.	Projection	Outcome	Prob.	Total prob.
	M_1			M_4			M_3	
0	$\frac{1}{2}$	$\lvert 0\rangle \otimes \lvert 1\rangle$	0	$\lvert\langle 1\lvert a\rangle\rvert^2$	$\alpha_1\lvert 0\rangle \otimes \lvert a\rangle$	**1**	$\lvert\langle a\lvert 1\rangle\rvert^2$	$\frac{\lvert\langle a\lvert 1\rangle\rvert^4}{2}$
0	$\frac{1}{2}$	$\lvert 0\rangle \otimes \lvert 1\rangle$	1	$\lvert\langle 1\lvert b\rangle\rvert^2$	$\beta_1\lvert 0\rangle \otimes \lvert b\rangle$	**1**	$\lvert\langle b\lvert 1\rangle\rvert^2$	$\frac{\lvert\langle b\lvert 1\rangle\rvert^4}{2}$
1	$\frac{1}{2}$	$-\lvert 1\rangle \otimes \lvert 0\rangle$	0	$\lvert\langle 0\lvert a\rangle\rvert^2$	$-\alpha_0\lvert 1\rangle \otimes \lvert a\rangle$	**0**	$\lvert\langle a\lvert 0\rangle\rvert^2$	$\frac{\lvert\langle a\lvert 0\rangle\rvert^4}{2}$
1	$\frac{1}{2}$	$-\lvert 1\rangle \otimes \lvert 0\rangle$	1	$\lvert\langle 0\lvert b\rangle\rvert^2$	$-\beta_0\lvert 1\rangle \otimes \lvert b\rangle$	**0**	$\lvert\langle b\lvert 0\rangle\rvert^2$	$\frac{\lvert\langle b\lvert 0\rangle\rvert^4}{2}$
	M_1			M_4			M_4	
0	$\frac{1}{2}$	$\lvert 0\rangle \otimes \lvert 1\rangle$	0	$\lvert\langle 1\lvert a\rangle\rvert^2$	$\alpha_1\lvert 0\rangle \otimes \lvert a\rangle$	**1**	0	
0	$\frac{1}{2}$	$\lvert 0\rangle \otimes \lvert 1\rangle$	1	$\lvert\langle 1\lvert b\rangle\rvert^2$	$\beta_1\lvert 0\rangle \otimes \lvert b\rangle$	**1**	1	$\frac{\lvert\langle 1\lvert b\rangle\rvert^2}{2}$
1	$\frac{1}{2}$	$-\lvert 1\rangle \otimes \lvert 0\rangle$	0	$\lvert\langle 0\lvert a\rangle\rvert^2$	$-\alpha_0\lvert 1\rangle \otimes \lvert a\rangle$	**0**	1	$\frac{\lvert\langle 0\lvert a\rangle\rvert^2}{2}$
1	$\frac{1}{2}$	$-\lvert 1\rangle \otimes \lvert 0\rangle$	1	$\lvert\langle 0\lvert b\rangle\rvert^2$	$-\beta_0\lvert 1\rangle \otimes \lvert b\rangle$	**0**	0	
	M_2			M_3			M_4	
0	$\frac{1}{2}$	$c\lvert a\rangle \otimes \lvert b\rangle$	0	$\lvert\langle b\lvert 0\rangle\rvert^2$	$c\overline{\beta}_0\lvert a\rangle \otimes \lvert 0\rangle$	**1**	$\lvert\langle 0\lvert b\rangle\rvert^2$	$\frac{\lvert\langle b\lvert 0\rangle\rvert^4}{2}$
0	$\frac{1}{2}$	$c\lvert a\rangle \otimes \lvert b\rangle$	1	$\lvert\langle b\lvert 1\rangle\rvert^2$	$c\overline{\beta}_1\lvert a\rangle \otimes \lvert 1\rangle$	**1**	$\lvert\langle 1\lvert b\rangle\rvert^2$	$\frac{\lvert\langle b\lvert 1\rangle\rvert^4}{2}$
1	$\frac{1}{2}$	$-c\lvert b\rangle \otimes \lvert a\rangle$	0	$\lvert\langle a\lvert 0\rangle\rvert^2$	$-c\overline{\alpha}_0\lvert b\rangle \otimes \lvert 0\rangle$	**0**	$\lvert\langle 0\lvert a\rangle\rvert^2$	$\frac{\lvert\langle a\lvert 0\rangle\rvert^4}{2}$
1	$\frac{1}{2}$	$-c\lvert b\rangle \otimes \lvert a\rangle$	1	$\lvert\langle a\lvert 1\rangle\rvert^2$	$-c\overline{\alpha}_1\lvert b\rangle \otimes \lvert 1\rangle$	**0**	$\lvert\langle 1\lvert a\rangle\rvert^2$	$\frac{\lvert\langle a\lvert 1\rangle\rvert^4}{2}$
	M_2			M_3			M_3	
0	$\frac{1}{2}$	$c\lvert a\rangle \otimes \lvert b\rangle$	0	$\lvert\langle b\lvert 0\rangle\rvert^2$	$c\overline{\beta}_0\lvert a\rangle \otimes \lvert b\rangle$	**1**	0	
0	$\frac{1}{2}$	$c\lvert a\rangle \otimes \lvert b\rangle$	1	$\lvert\langle b\lvert 1\rangle\rvert^2$	$c\overline{\beta}_1\lvert a\rangle \otimes \lvert 1\rangle$	**1**	1	$\frac{\lvert\langle 1\lvert b\rangle\rvert^2}{2}$
1	$\frac{1}{2}$	$-c\lvert b\rangle \otimes \lvert a\rangle$	0	$\lvert\langle a\lvert 0\rangle\rvert^2$	$-c\overline{\alpha}_0\lvert b\rangle \otimes \lvert 0\rangle$	**0**	1	$\frac{\lvert\langle 0\lvert a\rangle\rvert^2}{2}$
1	$\frac{1}{2}$	$-c\lvert b\rangle \otimes \lvert a\rangle$	1	$\lvert\langle a\lvert 1\rangle\rvert^2$	$-c\overline{\alpha}_1\lvert b\rangle \otimes \lvert 1\rangle$	**0**	0	

Adding up the probabilities we find

$$\frac{1}{8}\left(2+2\left(\lvert\langle a\lvert 0\rangle\rvert^2+\lvert\langle b\lvert 1\rangle\rvert^2\right)+\left(\lvert\langle a\lvert 0\rangle\rvert^4+\lvert\langle a\lvert 1\rangle\rvert^4+\lvert\langle b\lvert 0\rangle\rvert^4+\lvert\langle b\lvert 1\rangle\rvert^4\right)\right).$$