UNIVERSITEIT VAN JOHANNESBURG

FAKULTEIT NATUURWETENSKAP

DEPARTEMENT TOEGEPASTE WISKUNDE

MODULE **APM3B10**
KWANTUMBEREKENINGE

KAMPUS **APK**

EKSAMEN **NOVEMBER 2008**

DATUM: 20/11/2008 **SESSION:** 09:00 – 12:00

ASSESSOR DR. Y. HARDY

EKSTERNE MODERATOR DR. F. SOLMS

TYDSDUUR: 3 UUR **PUNTE:** 50

AANTAL BLADSYE: 3 BLADSYE

INSTRUKSIES: BEANTWOORD AL DIE VRAE

ALLE BEREKENINGS MOET GETOON WORD

SAKREKENAARS MAG GEBRUIK WORD

ALLE HOEKE WORD IN RADIALE GEMEET

DIE VOORGESKREWE HANDBOEK WORD TOEGELAAT

See reverse for English

VRAAG 1

(a) Is die matriks

$$\rho := \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$

'n digtheidsmatriks? (5)

(b) Is die matriks

$$\rho_2 := \sum_{j=1}^{2} (\mathbf{e}_j \otimes I_2)^T \rho (\mathbf{e}_j \otimes I_2)$$

waar

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

'n digtheidsmatriks? (5)

(10)

VRAAG 2

(a) Laat $u \in \mathbb{R}$, $u \neq 0$. Wys dat

$$\| u\mathbf{x} - \mathbf{a} \otimes \mathbf{b} \|^2 = |u|^2 \left\| \mathbf{x} - \left(\frac{1}{u}\mathbf{a} \right) \otimes \mathbf{b} \right\|^2, \qquad \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^{2n}$$

waar $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}}$ die Euklidiese norm vir $\mathbf{x} \in \mathbb{R}^{2n}$ is. (5)

(b) Bepaal die minimum van

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \mathbf{a} \otimes \mathbf{b} \right\|^2, \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad a_1, a_2, b_1, b_2 \in \mathbb{R}$$

waar $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}}$ die Euklidiese norm vir $\mathbf{x} \in \mathbb{R}^4$ is. (5)

(c) Bepaal die minimum van

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \mathbf{a} \otimes \mathbf{b} \right\|^2, \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad a_1, a_2, b_1, b_2 \in \mathbb{R}$$

waar $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}}$ die Euklidiese norm vir $\mathbf{x} \in \mathbb{R}^4$ is. (5)

(15)

VRAAG 3

Laat

$$\{\, |0\rangle, |1\rangle, |2\rangle, |3\rangle \,\}$$

'n ortonormale basis in $\mathbb{C}^4$ wees. Pas die kwantum Fourier-transformasie op $\mathbb{C}^4$ toe op die toestand

$$|\psi_1\rangle := \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |0\rangle + |3\rangle \otimes |1\rangle),$$

en op die toestand

$$|\psi_2\rangle := \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |1\rangle + |3\rangle \otimes |0\rangle)$$

d.w.s. pas $U_{QFT,4} \otimes I_4$ toe. Die kwantum Fourier-transformasie in $\mathbb{C}^4$ is

$$U_{QFT,4} = \frac{1}{2}\sum_{j=0}^{3}\sum_{k=0}^{3} e^{-i2\pi jk/4}|j\rangle\langle k|.$$

Gebruik u antwoorde om die periodisiteit van die ry van toestande in $|\psi_1\rangle$ (0101) en $|\psi_2\rangle$ (0110) te analiseer. **(10)**

VRAAG 4

Laat $\{\,|0\rangle, |1\rangle\,\}$ en $\{\,|a\rangle, |b\rangle\,\}$ twee ortonormale basisse in $\mathbb{C}^2$ wees. Gebruik die uitbreidings

$$|0\rangle = \langle a|0\rangle|a\rangle + \langle b|0\rangle|b\rangle, \qquad |1\rangle = \langle a|1\rangle|a\rangle + \langle b|1\rangle|b\rangle$$

om die ortonormaliteit van $\{\,|0\rangle, |1\rangle\,\}$ en die toestand

$$|\psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

in terme van $|a\rangle$ en $|b\rangle$ uit te druk.

Beskou die 4 meetbare kwantiteite

$$M_1 = |1\rangle\langle 1| \otimes I_2, \qquad M_2 = |b\rangle\langle b| \otimes I_2, \qquad M_3 = I_2 \otimes |1\rangle\langle 1|, \qquad M_4 = I_2 \otimes |b\rangle\langle b|$$

elkeen met uitkomstes 0 en 1. Vir elke ry van metings onder, bepaal die waarskynlikheid dat die uitkomste van die eerste en laaste meting verskillend is.

1.	M_1, M_3, M_3	5.	M_1, M_4, M_3
2.	M_1, M_3, M_4	6.	M_1, M_4, M_4
3.	M_2, M_4, M_4	7.	M_2, M_3, M_4
4.	M_2, M_4, M_3	8.	M_2, M_3, M_3

Elke ry metings bo word uitgevoer met waarskynlikheid $\frac{1}{8}$. Wat is die totale waarskynlikheid dat die uitkomstes van die eerste en laaste metings verskillend is? **(15)**

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UNIVERSITY OF JOHANNESBURG

FACULTY OF SCIENCE

DEPARTMENT OF APPLIED MATHEMATICS

MODULE	**APM3B10** QUANTUM COMPUTING
CAMPUS	**APK**
EXAM	**NOVEMBER 2008**

DATE: 20/11/2008 **SESSION:** 09:00 – 12:00

ASSESSOR DR. Y. HARDY

EXTERNAL MODERATOR DR. F. SOLMS

DURATION: 3 HOURS **MARKS:** 50

NUMBER OF PAGES: 3 PAGES

INSTRUCTIONS: ANSWER ALL THE QUESTIONS

ALL CALCULATIONS MUST BE SHOWN

POCKET CALCULATORS ARE PERMITTED

ALL ANGLES ARE MEASURED IN RADIANS

THE PRESCRIBED TEXT BOOK IS ALLOWED

Sien keersy vir Afrikaans

QUESTION 1

(a) Is the matrix

$$\rho := \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$

a density matrix? (5)

(b) Is the matrix

$$\rho_2 := \sum_{j=1}^{2} (\mathbf{e}_j \otimes I_2)^T \rho (\mathbf{e}_j \otimes I_2)$$

where

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

a density matrix? (5)

(10)

QUESTION 2

(a) Let $u \in \mathbb{R}$, $u \neq 0$. Show that

$$\|u\mathbf{x} - \mathbf{a} \otimes \mathbf{b}\|^2 = |u|^2 \left\| \mathbf{x} - \left(\frac{1}{u}\mathbf{a}\right) \otimes \mathbf{b} \right\|^2, \qquad \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \mathbf{x} \in \mathbb{R}^{2n}$$

where $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}}$ denotes the Euclidean norm for $\mathbf{x} \in \mathbb{R}^{2n}$. (5)

(b) Determine the minimum of

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \mathbf{a} \otimes \mathbf{b} \right\|^2, \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad a_1, a_2, b_1, b_2 \in \mathbb{R}$$

where $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}}$ denotes the Euclidean norm for $\mathbf{x} \in \mathbb{R}^4$. (5)

(c) Determine the minimum of

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \mathbf{a} \otimes \mathbf{b} \right\|^2, \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad a_1, a_2, b_1, b_2 \in \mathbb{R}$$

where $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}}$ denotes the Euclidean norm for $\mathbf{x} \in \mathbb{R}^4$. (5)

(15)

QUESTION 3

Let

$$\{\, |0\rangle,\ |1\rangle,\ |2\rangle,\ |3\rangle \,\}$$

denote an orthonormal basis in $\mathbb{C}^4$. Apply the quantum Fourier transform on $\mathbb{C}^4$ to the state

$$|\psi_1\rangle := \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |0\rangle + |3\rangle \otimes |1\rangle),$$

and to the state

$$|\psi_2\rangle := \frac{1}{2}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |1\rangle + |3\rangle \otimes |0\rangle)$$

i.e. apply $U_{QFT,4} \otimes I_4$. The quantum Fourier transform on $\mathbb{C}^4$ is given by

$$U_{QFT,4} = \frac{1}{2} \sum_{j=0}^{3} \sum_{k=0}^{3} e^{-i2\pi jk/4} |j\rangle\langle k|.$$

Use your answers to analyze the periodicity of the sequence of states in $|\psi_1\rangle$ (0101) and $|\psi_2\rangle$ (0110). **(10)**

QUESTION 4

Let $\{\, |0\rangle, |1\rangle \,\}$ and $\{\, |a\rangle, |b\rangle \,\}$ denote two orthonormal bases in $\mathbb{C}^2$. Use the expansions

$$|0\rangle = \langle a|0\rangle |a\rangle + \langle b|0\rangle |b\rangle, \qquad |1\rangle = \langle a|1\rangle |a\rangle + \langle b|1\rangle |b\rangle$$

to express the orthonormality of $\{\, |0\rangle, |1\rangle \,\}$ and to express

$$|\psi\rangle := \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$$

in terms of $|a\rangle$ and $|b\rangle$.

Consider the 4 observables

$$M_1 = |1\rangle\langle 1| \otimes I_2, \qquad M_2 = |b\rangle\langle b| \otimes I_2, \qquad M_3 = I_2 \otimes |1\rangle\langle 1|, \qquad M_4 = I_2 \otimes |b\rangle\langle b|$$

each with measurement outcomes 0 and 1. For each sequence of measurements below, determine the probability that the measurement outcomes of the first and last measurement are different.

1. M_1, M_3, M_3
2. M_1, M_3, M_4
3. M_2, M_4, M_4
4. M_2, M_4, M_3
5. M_1, M_4, M_3
6. M_1, M_4, M_4
7. M_2, M_3, M_4
8. M_2, M_3, M_3

Each measurement sequence above is performed with probability $\frac{1}{8}$. What is the total probability that the first and last measurement outcomes are different? **(15)**

END OF QUESTION PAPER