TEST IX (TOETS IX) 3B

1. We have $A(0) = I$ and $A(1) = U_{NOT}$. Thus A_0 has eigenvalues 1 and 1, and $A(1)$ has eigenvalues 1 and -1. We tabulate the eigenvalues and corresponding eigenvectors of $A(x)$

Eigenvalue	Eigenvector
1	$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$
$(-1)^x$	$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$

$$
\begin{aligned}
U_f &= (1-f(0))|00\rangle\langle 00| + f(0)|01\rangle\langle 00| \\
&+ f(0)|00\rangle\langle 01| + (1-f(0))|01\rangle\langle 01| \\
&+ (1-f(1))|10\rangle\langle 10| + f(1)|11\rangle\langle 10| \\
&+ f(1)|10\rangle\langle 11| + (1-f(1))|11\rangle\langle 11| \\
&= |0\rangle\langle 0| \otimes ((1-f(0))|0\rangle\langle 0| + f(0)|1\rangle\langle 0|) \\
&+ |0\rangle\langle 0| \otimes (f(0)|0\rangle\langle 1| + (1-f(0))|1\rangle\langle 1|) \\
&+ |1\rangle\langle 1| \otimes ((1-f(1))|0\rangle\langle 0| + f(1)|1\rangle\langle 0|) \\
&+ |1\rangle\langle 1| \otimes (f(1)|0\rangle\langle 1| + (1-f(1))|1\rangle\langle 1|) \\
&= |0\rangle\langle 0| \otimes ((1-f(0))(|0\rangle\langle 0| + |1\rangle\langle 1|) + f(0)(|0\rangle\langle 1| + |1\rangle\langle 0|)) \\
&+ |1\rangle\langle 1| \otimes ((1-f(1))(|0\rangle\langle 0| + |1\rangle\langle 1|) + f(1)(|0\rangle\langle 1| + |1\rangle\langle 0|)) \\
&= |0\rangle\langle 0| \otimes A(f(0)) + |1\rangle\langle 1| \otimes A(f(1))
\end{aligned}
$$

$$
\begin{aligned}
U_f\left(I \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) &= |0\rangle\langle 0| \otimes A(f(0))\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&+ |1\rangle\langle 1| \otimes A(f(1))\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&= |0\rangle\langle 0| \otimes (-1)^{f(0)}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&+ |1\rangle\langle 1| \otimes (-1)^{f(1)}\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&= (-1)^{f(0)}|0\rangle\langle 0| \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\
&+ (-1)^{f(1)}|1\rangle\langle 1| \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&= \left((-1)^{f(0)}\left(|0\rangle\langle 0| + (-1)^{f(0)+f(1)}|1\rangle\langle 1|\right)\otimes I\right) \\
&\cdot \left(I \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)
\end{aligned}
$$

Thus when $f(0) = f(1)$ we apply the identity operator to the first qubit and when $f(0) \neq f(1)$ we apply a phase change to the first qubit. The eigenvalues $(-1)^{f(0)}$ and $(-1)^{f(1)}$ are said to *kick back* to the first qubit. Remember that a phase change combined with two Hadamard transforms in the appropriate order implements a NOT gate.

2. a) Let $|\psi\rangle := a|0\rangle + b|1\rangle$.

$$
|\psi\rangle \otimes \frac{1}{\sqrt{2}}(I\otimes U)(|00\rangle + |11\rangle)\otimes|\phi\rangle
$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{2}}(|00\rangle + |11\rangle)\otimes U(a|0\rangle + b|1\rangle)\otimes|\phi\rangle \\
&+ \frac{1}{2\sqrt{2}}(|00\rangle - |11\rangle)\otimes U(a|0\rangle - b|1\rangle)\otimes|\phi\rangle \\
&+ \frac{1}{2\sqrt{2}}(|01\rangle + |10\rangle)\otimes U(a|1\rangle + b|0\rangle)\otimes|\phi\rangle \\
&+ \frac{1}{2\sqrt{2}}(|01\rangle - |10\rangle)\otimes U(a|1\rangle - b|0\rangle)\otimes|\phi\rangle \\
&= |\phi\rangle
\end{aligned}
$$

We measure in the Bell basis

$$
\left\{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\right\}.
$$

From $|\phi\rangle$ we can see that the first two qubits are in each of the Bell states with equal probability. Thus if we measure we obtain a result corresponding to each of the Bell states and can perform a transform to obtain $U|\psi\rangle \otimes |\phi\rangle$ in the last two qubits as follows

Bell State	Transform
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	I
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$U\left((\|0\rangle\langle 0\| - \|1\rangle\langle 1\|)\otimes I\right)U^*$
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$U(U_{NOT}\otimes I)U^*$
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$U\left((\|0\rangle\langle 1\| - \|1\rangle\langle 0\|)\otimes I\right)U^*$

Thus after measurement and applying the corresponding transform we obtain $U|\psi\rangle \otimes |\phi\rangle$ as the last two qubits. Thus if Alice and Bob initially share the entangled pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and Bob applied U to his two qubits. Alice can perform a measurement in the Bell basis on her qubit and her part of the entangled pair and send the result (two bits) to Bob who applies the corresponding transform to his part of the entangled pair. Thus with probability $\frac{1}{4}$ Bob can begin the computation $U|\psi\rangle \otimes |\phi\rangle$ without knowing the state $|\psi\rangle$ and still obtain the correct result after Alice measures her two qubits. With probability $\frac{3}{4}$ he still has to apply a transform which is independent of $|\psi\rangle$.

b) Alice teleports $|\psi\rangle$ to Bob with one entangle pair, Bob performs the computation $U|\psi\rangle \otimes |\phi\rangle$ on his two qubits and then teleports the first qubit back to Alice with a second entangled pair. Thus 4 bits of communication are used in this scheme (Alice sends two to Bob, and then Bob two to Alice). Alice and Bob can perform U_{CNOT} even though their qubits are spatially separated if they have prior entanglement.