TEST VIII (TOETS VIII) 3B

1. Obviously $x+y+z$ must be even. Thus the sum includes only an even number (0 or 2) of odd numbers. Thus we have the combinations

   $$\{ (0,0,0),(0,1,1),(0,0,2),(1,1,2),(0,2,2),(0,1,3),(2,2,2),(1,2,3),(0,3,3) \},$$

   – (x,y,z) is an element of the set of all permuatations of elements of the above set. When $x+y+z$ is even, $(x+y+z)$ mod $4 \in \{0,2\}$. Now when $x+y+z = 0$ mod 2 then $f(x,y,z) \in \{0,1\}$. Since $x+y+z = 0$ mod 2 the least significant bit of the sum must be zero. The least significant bit is given by $x_0 \oplus y_0 \oplus z_0 = 0$. Thus we have

   $$(x_0,y_0,z_0) \in \{ (0,0,0),(0,1,1),(1,0,1),(1,1,0) \}.$$

   Note the symmetry in the state $|\psi\rangle$. A qubit can be exchanged with any other without our being able to determine any difference. Thus we need only calculate the transform for $(0,0,0)$ and $(0,1,1)$. For $(0,0,0)$ we have $f_3(0,0,0)|\psi\rangle = I \otimes I \otimes I|\psi\rangle = |\psi\rangle$. Measuring the qubits yields

   $$(s_x,s_y,s_z) \in \{ (0,0,0),(0,1,1),(1,0,1),(1,1,0) \}$$

   with equal probability. In each case we find $s_x+s_y+s_z = 0$ mod 2. For $(0,1,1)$ we have $f_3(0,1,1) = I \otimes U_H \otimes U_H$. Note that

   $$|\psi\rangle = \frac{1}{2}|0\rangle \otimes (|00\rangle - |11\rangle) - \frac{1}{2}|1\rangle \otimes (|01\rangle + |10\rangle).$$

   Thus

   $$f_3(0,1,1)|\psi\rangle = \frac{1}{2}|0\rangle \otimes (|01\rangle + |10\rangle) - \frac{1}{2}|1\rangle \otimes (|00\rangle - |11\rangle).$$

   We find that measuring the qubits yields

   $$(s_x,s_y,s_z) \in \{ (0,1,0),(1,0,0),(0,0,1),(1,1,1) \}$$

   with equal probability. In each case we find $s_x+s_y+s_z = 1$ mod 2.

(x_0,y_0,z_0)	$x_0+y_0+z_0$	$s_x+s_y+s_z$ mod 2
(0,0,0)	0	0
(0,1,1)	1	1
(1,0,1)	1	1
(1,1,0)	1	1

We find that $(s_x + s_y + s_z \; mod \; 2) = x_0 + y_0 + z_0$.
Thus for three parties to calculate $f(x,y,z)$, where each party has one of the x, y and z, it is sufficient for each party to send one bit ($x_1 \oplus s_x$ or $y_1 \oplus s_y$ or $z_1 \oplus s_z$) to the other parties to calculate $f(x,y,z)$. In other words each party can calculate

$$x_1 \oplus s_x \oplus y_1 \oplus s_y \oplus y_1 \oplus s_y \oplus z_1 \oplus s_z = x_1 \oplus y_1 \oplus z_1 \oplus (x_0 + y_0 + z_0) = f(x,y,z)$$

after communication. In other words three bits broadcast to all parties are sufficient to calculate $f(x,y,z)$. Classically it is necessary that 4 bits be broadcast.

2.

$$|\phi\rangle = \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle)$$

$$\begin{aligned}
& \frac{1}{2\sqrt{2}}(|00\rangle + |11\rangle) \otimes (a|0\rangle + b|1\rangle) \\
+ \quad & \frac{1}{2\sqrt{2}}(|00\rangle - |11\rangle) \otimes (a|0\rangle - b|1\rangle) \\
+ \quad & \frac{1}{2\sqrt{2}}(|01\rangle + |10\rangle) \otimes (a|1\rangle + b|0\rangle) \\
+ \quad & \frac{1}{2\sqrt{2}}(|01\rangle - |10\rangle) \otimes (a|1\rangle - b|0\rangle) \\
= \quad & \frac{1}{2\sqrt{2}}(a|000\rangle + a|110\rangle + b|001\rangle + b|111\rangle) \\
+ \quad & \frac{1}{2\sqrt{2}}(a|000\rangle - a|110\rangle - b|001\rangle + b|111\rangle) \\
+ \quad & \frac{1}{2\sqrt{2}}(a|011\rangle + a|101\rangle + b|010\rangle + b|100\rangle) \\
+ \quad & \frac{1}{2\sqrt{2}}(a|011\rangle - a|101\rangle - b|010\rangle + b|100\rangle) \\
= \quad & \frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|100\rangle + b|111\rangle) \\
= \quad & |\phi\rangle
\end{aligned}$$

We measure in the Bell basis

$$\left\{ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),\ \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \right\}.$$

From $|\phi\rangle$ we can see that the first two qubits are in each of the Bell states with equal probability. Thus if we measure we obtain a result corresponding to each of the Bell states and can perform a transform to obtain $|\psi\rangle$ in the last qubit as follows

Bell State	Transform
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	I
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	$\|0\rangle\langle 0\| - \|1\rangle\langle 1\|$
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	U_{NOT}
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\|0\rangle\langle 1\| - \|1\rangle\langle 0\|$

Thus after measurement and applying the corresponding transform we obtain $|\psi\rangle$ as the last qubit. Thus if Alice and Bob initially share the entangled pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, Alice can perform a measurement in the Bell basis on her qubit and her part of the entangled pair and send the result (two bits) to Bob who applies the corresponding transform to his part of the entangled pair. Thus the state $|\psi\rangle$ is teleported from Alice's qubit to Bob's qubit. Note that the Bell basis is obtained by applying $U_{CNOT}(U_H \otimes I)$ to the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The transform is obviously invertible, thus we can also measure the first two qubits in the computational basis after applying $(U_H \otimes I)U_{CNOT}$ (i.e. teleportation does not depend on the ability to measure with respect to the Bell basis).