TEST VI (TOETS VI) 3B

1.

$$
\begin{aligned}
e^{tx^2\frac{d}{dx}}x &= x + tx^2 + \frac{2t^2x^3}{2!} + \frac{3\cdot 2\cdot t^3x^4}{3!} + \dots \\
&= x(1 + tx + t^2x^2 + t^3x^3 + \dots) \\
&= x(1 + (tx) + (tx)^2 + (tx)^3 + \dots) \\
&= \frac{x}{1-tx}
\end{aligned}
$$

where we used the geometric series for $|u| < 1$

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

For the differential equation

$$\frac{dx}{dt} = x^2 \Rightarrow \int x^{-2}dx = \int dt \Rightarrow -x^{-1} = t + c,$$

$$t = 0 \Rightarrow x = x_0 \Rightarrow c = -x_0.$$

Thus

$$
\begin{aligned}
x^{-1} &= -t + x_0^{-1} \\
x(t) &= \frac{1}{x_0^{-1} - t} = \frac{x_0}{1 - tx_0}
\end{aligned}
$$

The solution is also given by

$$e^{tx^2\frac{d}{dx}}x\Big|_{x=x_0}.$$

2. Let $a \in \{0, 1\}$.

$$
\begin{aligned}
U_H A U_H|a\rangle &= \frac{1}{\sqrt{2}}U_H A(|0\rangle + (-1)^a|1\rangle) \\
&= \frac{1}{\sqrt{2}}U_H(|0\rangle + (-1)^{a+1}|1\rangle) \\
&= \frac{1}{\sqrt{2}}U_H(|0\rangle + (-1)^{\overline{a}}|1\rangle) \\
&= |\overline{a}\rangle.
\end{aligned}
$$

In other words $U_H A U_H = U_{NOT}$, $U_H A U_H|0\rangle = |1\rangle$, $U_H A U_H|1\rangle = |0\rangle$.

$$
\begin{aligned}
&(U_H \otimes U_H)U_{CNOT}(U_H \otimes U_H)|a,b\rangle \\
&= \frac{1}{2}(U_H \otimes U_H)U_{CNOT}\Big((|0\rangle + (-1)^a|1\rangle) \otimes (|0\rangle + (-1)^b|1\rangle)\Big) \\
&= \frac{1}{2}(U_H \otimes U_H)(|00\rangle + (-1)^b|01\rangle + (-1)^a|11\rangle + (-1)^{a+b}|10\rangle) \\
&= \frac{1}{2}(U_H \otimes U_H)(|0\rangle \otimes (|0\rangle + (-1)^b|1\rangle) + (-1)^a(|1\rangle \otimes (|1\rangle + (-1)^b|0\rangle) \\
&= \frac{1}{2}(U_H \otimes U_H)(|0\rangle \otimes (|0\rangle + (-1)^b|1\rangle) + (-1)^{a+b}(|1\rangle \otimes (|0\rangle + (-1)^b|1\rangle) \\
&= \frac{1}{2}(U_H \otimes U_H)(|0\rangle \otimes (|0\rangle + (-1)^b|1\rangle) + (-1)^{a+b}(|1\rangle \otimes (|0\rangle + (-1)^b|1\rangle) \\
&= \frac{1}{2}(U_H \otimes U_H)(|0\rangle + (-1)^{a+b}|1\rangle) \otimes (|0\rangle + (-1)^b|1\rangle) \\
&= |a \oplus b, b\rangle
\end{aligned}
$$

In other words we have the controlled NOT where the control qubit is the second qubit and the target qubit the first qubit.

3.

$$
\begin{aligned}
\Big[\, |0\rangle\langle 1|\,,\, |1\rangle\langle 0| \,\Big] &= |0\rangle\langle 0| - |1\rangle\langle 1| \\
\exp(t|0\rangle\langle 1|) &= \sum_{j=0}^{\infty} \frac{t^j}{j!}(|0\rangle\langle 1|)^j = I + t|0\rangle\langle 1| \\
\exp(t|1\rangle\langle 0|) &= \sum_{j=0}^{\infty} \frac{t^j}{j!}(|1\rangle\langle 0|)^j = I + t|1\rangle\langle 0| \\
\exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|) &= I + t|0\rangle\langle 1| + t|1\rangle\langle 0| + t^2|0\rangle\langle 0| \\
\exp(t|1\rangle\langle 0|)\exp(t|0\rangle\langle 1|) &= I + t|0\rangle\langle 1| + t|1\rangle\langle 0| + t^2|1\rangle\langle 1| \\
(|0\rangle\langle 1| + |1\rangle\langle 0|)^2 &= I \\
\exp(t|0\rangle\langle 1| + t|1\rangle\langle 0|) &= \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!} I + \sum_{j=0}^{\infty} \frac{t^{2j+1}}{(2j+1)!}(|0\rangle\langle 1| + |1\rangle\langle 0|) \\
&= \cosh(t) I + \sinh(t)(|0\rangle\langle 1| + |1\rangle\langle 0|) \\
&\neq \exp(t|0\rangle\langle 1|)\exp(t|1\rangle\langle 0|)
\end{aligned}
$$

4.

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow 1 \cdot 1 = 1 \cdot 1 \Rightarrow separable$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow 1 \cdot (-1) \neq 1 \cdot 1 \Rightarrow entangled$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow 1 \cdot (-1) = 1 \cdot (-1) \Rightarrow separable$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow 1 \cdot (-1) \neq (-1) \cdot (-1) \Rightarrow entangled$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow 1 \cdot 1 \neq 0 \cdot 0 \Rightarrow entangled$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow 1 \cdot 0 = 0 \cdot 1 \Rightarrow separable$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow 0 \cdot (-1) = 1 \cdot 0 \Rightarrow separable$$

$$\frac{1}{2}\begin{pmatrix} i \\ 1 \\ -i \\ i \end{pmatrix} \Rightarrow i \cdot i \neq 1 \cdot i \Rightarrow entangled.$$