TEST V (TOETS V) 3B

1. The eigenvalue equation is

$$\frac{d^2}{dx^2} f = \lambda f.$$

This is a second order linear homogeneous differential equation. We use the exponential ansatz $f(x) = e^{mx}$, thus $m^2 = \lambda$. We have the solution

$$f(x) = Ae^{\sqrt{\lambda}x} + x^{\delta_{0,\lambda}} Be^{-\sqrt{\lambda}x}.$$

The eigenvalues are the complex numbers $\mathbf{C}$. Since

$$B = \{u_n^{(+)}, u_n^{(-)} \quad n \in \mathbf{N}\}$$

$$u_n^{(+)} = \frac{1}{\sqrt{a}} \cos\left(\frac{(n-\frac{1}{2})\pi q}{a}\right), \qquad u_n^{(-)} = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi q}{a}\right)$$

is an orthonormal basis in $L_2(-a, a)$ (and also the eigenfunctions of $\hat{H}$) and $\psi(q, 0) \in B$, $\psi(q, 0)$ expands to itself.

$$\begin{aligned}
\psi(q,t) = e^{-i\hat{H}t/\hbar}\psi(q,0) &= \sum_{j=0}^{\infty} \frac{(\frac{it\hbar}{2m}\frac{d^2}{dq^2})^j}{j!} \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right) \\
&= \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right) \left(\sum_{j=0}^{\infty} \frac{(-\frac{it\hbar\pi^2}{2ma^2})^j}{j!}\right) \\
&= \frac{1}{\sqrt{a}} \sin\left(\frac{\pi q}{a}\right) e^{-\frac{it\hbar\pi^2}{2ma^2}}.
\end{aligned}$$

$$\begin{aligned}
\langle \phi, \psi(q,t) \rangle &= \int_{-a}^{a} \frac{1}{a} \sin^2\left(\frac{\pi q}{a}\right) e^{\frac{it\hbar\pi^2}{2ma^2}} dq \\
&= \int_{-a}^{a} \frac{1}{a} \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi q}{a}\right)\right) e^{\frac{it\hbar\pi^2}{2ma^2}} dq \\
&= \left(\frac{q}{2a} - \frac{1}{4\pi}\sin\left(\frac{2\pi q}{a}\right)\right) e^{\frac{it\hbar\pi^2}{2ma^2}} \Big|_{-a}^{a} \\
&= e^{\frac{it\hbar\pi^2}{2ma^2}}.
\end{aligned}$$

Thus

$$P = |e^{\frac{it\hbar\pi^2}{2ma^2}}|^2 = 1.$$

2.

$$\mathbf{x}\mathbf{x}^T = \begin{pmatrix} \cos^2\phi_1\cos^2\phi_2 & \cos^2\phi_1\sin\phi_2\cos\phi_2 & \sin\phi_1\cos\phi_1\cos\phi_2 \\ \cos^2\phi_1\sin\phi_2\cos\phi_2 & \sin^2\phi_2\cos^2\phi_1 & \sin\phi_1\cos\phi_1\sin\phi_2 \\ \cos\phi_1\sin\phi_1\cos\phi_2 & \sin\phi_1\cos\phi_1\sin\phi_2 & \sin^2\phi_1 \end{pmatrix}.$$

a) Clearly the matrix is symmetric and real and is thus hermitian. To determine if the matrix is unitary we must determine if $(\mathbf{x}\mathbf{x}^T)^T$ is the inverse of the above matrix. It is easy to verify that $\mathbf{x}$ is normalized.

$$\begin{aligned} (\mathbf{x}\mathbf{x}^T)^T\mathbf{x}\mathbf{x}^T &= \mathbf{x}(\mathbf{x}^T\mathbf{x})\mathbf{x}^T \\ &= \mathbf{x}\mathbf{x}^T \end{aligned}$$

No choice of $\phi_1, \phi_2 \in \mathbf{R}$ can yield the identity matrix. The matrix is not unitary.
b) Since $\mathbf{x}$ is normalized we immediately find the eigenvalue 1 with corresponding eigenvector $\mathbf{x}$

$$(\mathbf{x}\mathbf{x}^T)\mathbf{x} = \mathbf{x}(\mathbf{x}^T\mathbf{x}) = \mathbf{x}.$$

Since $(\mathbf{x}\mathbf{x}^T)^2 = \mathbf{x}\mathbf{x}^T$, we find that the eigenvalues λ satisfy $\lambda = \lambda^2$, i.e $\lambda = 0$ or $\lambda = 1$. We also note that $\mathrm{tr}(\mathbf{x}\mathbf{x}^T) = 1$. Thus the eigenvaules are 0, 0 and 1.

3. It turns out that the adjoint of an operator can be obtained by simply swapping the labels of the corresponding bra and ket vectors in the sum, and taking the conjugate of all complex coefficients, thus

$$A^* = \bar{i}|1\rangle\langle 0| - \bar{i}|0\rangle\langle 1|.$$

Formally we can determine A^* as follows. Let

$$A^* = a_{00}|0\rangle\langle 0| + a_{01}|0\rangle\langle 1| + a_{10}|1\rangle\langle 0| + a_{11}|1\rangle\langle 1|.$$

The bra vector corresponding to the ket $A|y\rangle$ is $\langle y|A^*$. We require that $\langle y|Ax\rangle = \langle A^*y|x\rangle$ (or $\langle y|(A|x\rangle) = (\langle y|A^*)|x\rangle)$ *for all* $|x\rangle = x_0|0\rangle + x_1|1\rangle$ *and* $|y\rangle = y_0|0\rangle + y_1|1\rangle$. We find ($A$ self adjoint)

$$\begin{aligned} A|x\rangle &= ix_1|0\rangle - ix_0|1\rangle \\ \langle y|A^* &= (\overline{y_0}a_{00} + \overline{y_1}a_{10})\langle 0| + (\overline{y_0}a_{01} + \overline{y_1}a_{11})\langle 1| \end{aligned}$$

$$
\begin{aligned}
\langle y|(A|x\rangle) &= ix_1\overline{y_0} - ix_0\overline{y_1} \\
(\langle y|A^*)|x\rangle &= x_0(\overline{y_0}a_{00} + \overline{y_1}a_{10}) + x_1(\overline{y_0}a_{01} + \overline{y_1}a_{11}) \\
i\overline{y_0} &= (\overline{y_0}a_{01} + \overline{y_1}a_{11}) \\
-i\overline{y_1} &= (\overline{y_0}a_{00} + \overline{y_1}a_{10}) \\
a_{00} &= 0 \\
a_{01} &= i \\
a_{10} &= -i \\
a_{11} &= 0
\end{aligned}
$$

The eigenvalue equation is

$$A(a|0\rangle + b|1\rangle) = \lambda(a|0\rangle + b|1\rangle).$$

Thus we have the equations

$$
\begin{aligned}
-ia &= \lambda b \\
ib &= \lambda a
\end{aligned}
$$

If $\lambda = 0$ we have $a = 0$ and $b = 0$. Thus we consider only $\lambda \neq 0$. Obviously we may use $b \neq 0$ (thus $a \neq 0$). We obtain

$$
\begin{aligned}
\lambda &= -\frac{ia}{b} \\
ib &= -\frac{ia}{b}a \\
ib^2 &= -ia^2 \\
b &= \pm ia
\end{aligned}
$$

Using $|a|^2 + |b|^2 = 1$ we find $|a| = \pm\frac{1}{\sqrt{2}}$. Thus we obtain the eigenvalues and corresponding orthonormal eigenvectors

$$\lambda = -1, \quad \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\lambda = 1, \quad \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle).$$

We determine A^n.

$$
\begin{aligned}
A^2 &= |0\rangle\langle 0| + |1\rangle\langle 1| = I \\
A^3 &= A \\
A^4 &= I
\end{aligned}
$$

Thus

$$A^n = \begin{cases} A & n \text{ odd} \\ I & n \text{ even} \end{cases} .$$

$$\begin{aligned}
U := \exp(-i\hat{H}t/\hbar) &= \sum_{j=0}^{\infty} \frac{(-\frac{it}{\hbar})^j A^j}{j!} \\
&= \sum_{j=0}^{\infty} \frac{(-\frac{it}{\hbar})^{2j}}{(2j)!} I + \sum_{j=0}^{\infty} \frac{(-\frac{it}{\hbar})^{2j+1}}{(2j+1)!} A \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{t}{\hbar})^{2j}}{(2j)!} I - i \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{it}{\hbar})^{2j+1}}{(2j+1)!} A \\
&= \cos(t/\hbar) I - i \sin(t/\hbar) A \\
&= \cos(t/\hbar)(|0\rangle\langle 0| + |1\rangle\langle 1|) + \sin(t/\hbar)(|0\rangle\langle 1| - |1\rangle\langle 0|).
\end{aligned}$$

For the NOT operation we can use $U(t = \pi\hbar/2) = |0\rangle\langle 1| - |1\rangle\langle 0|$. We can use any $t = (2k+1)\pi\hbar/2$, $k \in \mathbf{N}_0$.

$$\begin{aligned}
U(t = \pi\hbar/4) &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 1| \\
U(t = \pi\hbar/4)^2 &= U(t = \pi\hbar/2) = |0\rangle\langle 1| - |1\rangle\langle 0|.
\end{aligned}$$

Thus we find $U(t = \pi\hbar/4)^2 = U(t = \pi\hbar/2)$, i.e. $U(t = \pi\hbar/4)$ acts as the square root of our NOT operation. Traditionally in quantum computation we use $U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|$. In this case for the $\sqrt{\text{NOT}}$ operation we use $U_{\sqrt{NOT}} = \frac{1}{2}(1+i)(|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{2}(1-i)(|0\rangle\langle 1| + |1\rangle\langle 0|)$.