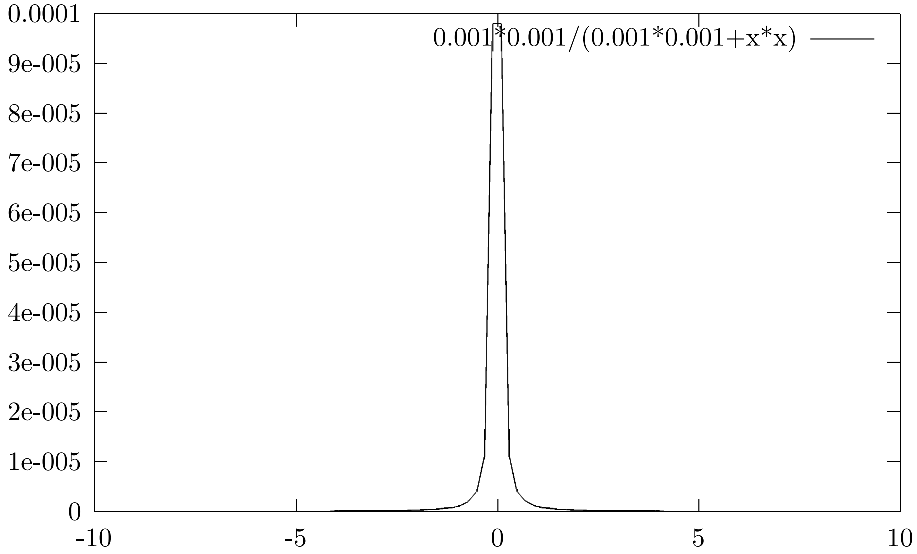
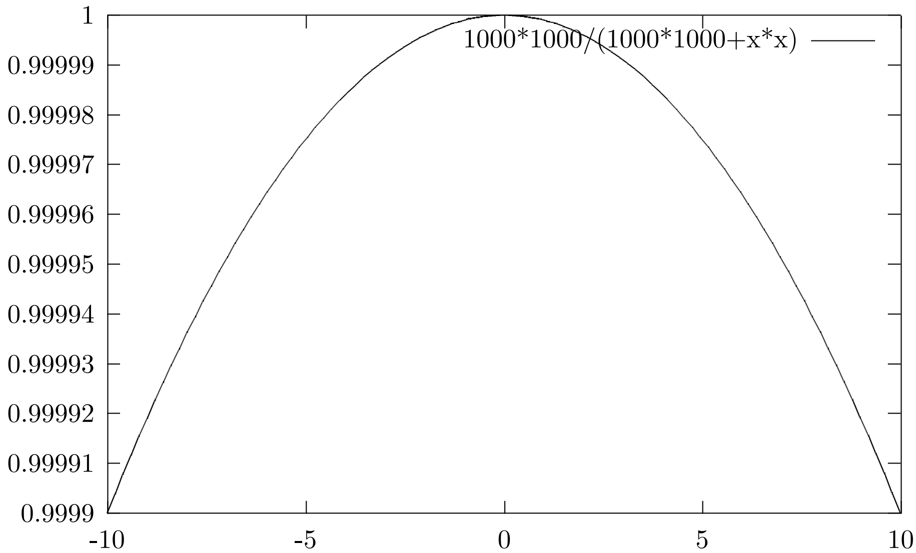
TEST IV (TOETS IV) 3B

1.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\alpha}{2} e^{ikx} e^{-\alpha|x|} dx &= \int_{-\infty}^{0} \frac{\alpha}{2} e^{(\alpha+ik)x} dx + \int_{0}^{\infty} \frac{\alpha}{2} e^{(-\alpha+ik)x} dx \\
&= \left[\frac{\alpha}{2(\alpha+ik)} e^{(\alpha+ik)x} \right]_{-\infty}^{0} + \left[\frac{\alpha}{2(-\alpha+ik)} e^{(-\alpha+ik)x} \right]_{0}^{\infty} \\
&= \frac{\alpha}{2(\alpha+ik)} - \frac{\alpha}{2(-\alpha+ik)} \\
&= \frac{\alpha^2}{\alpha^2+k^2}
\end{aligned}$$

The figures below illustrate for α small (0.001) and α large (1000).





$$\begin{aligned}
\int_{\infty}^{\infty} \frac{\alpha}{2} e^{-\alpha|x|} dx &= \int_{-\infty}^{0} \frac{\alpha}{2} e^{\alpha x} dx + \int_{0}^{\infty} \frac{\alpha}{2} e^{-\alpha x} dx \\
&= \left[\frac{\alpha e^{\alpha x}}{2}\right]_{-\infty}^{0} + \left[\frac{\alpha e^{-\alpha x}}{2}\right]_{0}^{\infty} \\
&= 1.
\end{aligned}$$

2. From the first condition

$$\begin{aligned}
\int_0^{2\pi} p(x)\cos(x)dx &= p(x)\sin(x)\Big|_0^{2\pi} - \int_0^{2\pi} (3a_3x^3 + 2a_2x + a_1)\sin(x)dx \\
&= (3a_3x^3 + 2a_2x + a1)\cos(x)\Big|_0^{2\pi} \\
&\quad - \int_0^{2\pi} (6a_3x + 2a_2x + a_1)\cos(x)dx \\
&= 12\pi^2 a_3 + 4\pi a_2 \\
&\quad -(6a_3x + 2a_2)\sin(x)\Big|_0^{2\pi} + \int_0^{2\pi} 6a_3\sin(x)dx \\
&= 12\pi^2 a_3 + 4\pi a_2 - (6a_3\cos(x))\Big|_0^{2\pi} \\
&= 12\pi^2 a_3 + 4\pi a_2 = 0 \\
a_2 &= -3\pi a_3. \qquad (1)
\end{aligned}$$

From the second condition we find

$$\frac{32\pi^5}{5}a_3 + \frac{16\pi^4}{4}a_2 + \frac{8\pi^3}{3}a_1 + \frac{4\pi^2}{2}a_0 = 0 \tag{2}$$

Inserting (1) yields

$$\frac{32}{5}\pi^3 a_3 - 12\pi^3 a_3 + \frac{8}{3}\pi a_1 + 2a_0 = 0$$

$$\begin{aligned} a_3 &= \frac{5}{28}(\frac{8}{3}\pi^{-2}a_1 + 2\pi^{-3}a_0) \\ a_2 &= -\frac{15}{28}(\frac{8}{3}\pi^{-1}a_1 + 2\pi^{-2}a_0) \end{aligned}$$

Selecting values for a_0 and a_1 gives a particular solution.

3. Let

$$A := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} = 2A.$$

Thus we have

$$A^{2n} = 2^{n-1}A^2, \qquad A^{2n+1} = 2^n A$$

and

$$\begin{aligned} \exp(-i\hat{S}_x t/\hbar) &= I + \sum_{j=0}^{\infty} \frac{(-it\omega/\sqrt{2})^{2j+1}}{(2j+1)!} 2^j A + \sum_{j=1}^{\infty} \frac{(-it\omega/\sqrt{2})^{2j}}{(2j)!} 2^{j-1} A^2 \\ &= I + \frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} \frac{(-it\omega)^{2j+1}}{(2j+1)!} A + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-it\omega)^{2j}}{(2j)!} A^2 \\ &= I + \frac{1}{\sqrt{2}} \sinh(-it\omega) A + \frac{1}{2}(\cosh(-it\omega) - 1) A^2 \\ &= I - \frac{i}{\sqrt{2}} \sin(t\omega) A + \frac{1}{2}(\cos(t\omega) - 1) A^2 \\ &= \begin{pmatrix} \frac{1}{2}\cos(t\omega) + \frac{1}{2} & -\frac{i}{\sqrt{2}}\sin(t\omega) & \frac{1}{2}\cos(t\omega) - \frac{1}{2} \\ -\frac{i}{\sqrt{2}}\sin(t\omega) & \cos(t\omega) & -\frac{i}{\sqrt{2}}\sin(t\omega) \\ \frac{1}{2}\cos(t\omega) - \frac{1}{2} & -\frac{i}{\sqrt{2}}\sin(t\omega) & \frac{1}{2}\cos(t\omega) + \frac{1}{2} \end{pmatrix}. \end{aligned}$$

$$\psi(t) = \exp(-i\hat{S}_x t/\hbar)\psi(0) = \frac{1}{\sqrt{3}}\begin{pmatrix} \cos(t\omega) - \frac{i}{\sqrt{2}}\sin(t\omega) \\ \cos(t\omega) - i\sqrt{2}\sin(t\omega) \\ \cos(t\omega) - \frac{i}{\sqrt{2}}\sin(t\omega) \end{pmatrix}.$$

The probability of finding $\psi(t)$ in the initial state is

$$\begin{aligned} |\langle\psi(t)|\psi(0)\rangle|^2 &= \frac{1}{9}|3\cos(t\omega) - i2\sqrt{2}\sin(t\omega)|^2 \\ &= \frac{1}{9}(9\cos^2(t\omega) + 8\sin^2(t\omega)) \\ &= \cos^2(t\omega) + \frac{8}{9}\sin^2(t\omega) \\ &= 1 - \frac{\sin^2(t\omega)}{9}. \end{aligned}$$