TEST III (TOETS III) 3B

1. (a)

$$\mathrm{tr}(AA^T) = \mathrm{tr}\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = 9.$$

Thus $||A|| = 3$.

(b) Using the method of the Lagrange multiplier we obtain

$$\begin{aligned} F(x_1, x_2, x_3) &= 3(x_1 + x_2 + x_3)^2 + \lambda(x_1^2 + x_2^2 + x_3^2) \\ \frac{\partial F}{\partial x_1} &= 0 = 6(x_1 + x_2 + x_3) + 2\lambda x_1 \qquad (1) \\ \frac{\partial F}{\partial x_2} &= 0 = 6(x_1 + x_2 + x_3) + 2\lambda x_2 \qquad (2) \\ \frac{\partial F}{\partial x_3} &= 0 = 6(x_1 + x_2 + x_3) + 2\lambda x_3 \qquad (3) \end{aligned}$$

If $\lambda = 0$ equations (1)–(3) give $x_1 + x_2 + x_3 = 0$, i.e. $||A\mathbf{x}|| = 0$. Suppose $\lambda \neq 0$, then (1)–(3) gives $x_1 = x_2 = x_3$. Two more solutions are then given by

$$x_1 = x_2 = x_3 = \pm\frac{1}{\sqrt{3}}.$$

Evaluating F for these solutions gives 9. The maximum is 9. Thus $||A|| = 3$.

(c) The eigenvalues of A are 0, 0 and 3.
The eigenvalues of AA^T are 0, 0 and 9.
(a) $||A||$ is the square root of the sum of the eigenvalues of AA^T.
(b) $||A||$ is the sum of the eigenvalues of A.

2. Let $f(x) = ax^3 + bx^2 + cx + d$. We obtain the equations

$$\begin{aligned} \frac{a}{5} + \frac{b}{4} + \frac{c}{3} + \frac{d}{2} &= 0 \\ \frac{a}{6} + \frac{b}{5} + \frac{c}{4} + \frac{d}{3} &= 0 \\ \frac{a}{7} + \frac{b}{6} + \frac{c}{5} + \frac{d}{4} &= 0 \end{aligned}$$

Solving for a, b and c in terms of d yields

$$a = -\frac{35}{4}d, \quad b = 15d, \quad c = -\frac{15}{2}d.$$

Selecting d (for example $d = 1$) gives a solution.

3.

$$\begin{aligned}
\langle f(x), \phi_0 \rangle &= \int_0^{\frac{1}{2}} x dx + \int_{\frac{1}{2}}^1 (1-x) = \frac{x^2}{2}\Big|_0^{\frac{1}{2}} + (x - \frac{x^2}{2})\Big|_{\frac{1}{2}}^1 = \frac{1}{4} \\
\text{for} \quad n \in \mathrm{N} \langle f(x), \phi_n \rangle &= \int_0^{\frac{1}{2}} \sqrt{2} x \cos(\pi n x) dx \\
&\quad + \int_{\frac{1}{2}}^1 (\sqrt{2}\cos(\pi n x) - \sqrt{2} x \cos(\pi n x)) dx \\
&= \frac{\sqrt{2}}{\pi n} x \sin(\pi n x)\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{\sqrt{2}}{\pi n} \sin(\pi n x) dx \\
&\quad + \frac{\sqrt{2}}{\pi n} \sin(\pi n x)\Big|_{\frac{1}{2}}^1 - \frac{\sqrt{2}}{\pi n} x \sin(\pi n x)\Big|_{\frac{1}{2}}^1 \\
&\quad + \int_{\frac{1}{2}}^1 \frac{\sqrt{2}}{\pi n} \sin(\pi n x) dx \\
&= \frac{1}{\sqrt{2}\pi n} \sin(\frac{\pi n}{2}) + \frac{\sqrt{2}}{\pi^2 n^2} \cos(\pi n x)\Big|_0^{\frac{1}{2}} \\
&\quad + \frac{\sqrt{2}}{\pi n} \sin(\pi n) - \frac{\sqrt{2}}{\pi n} \sin(\frac{\pi n}{2}) \\
&\quad - \frac{\sqrt{2}}{\pi n} \sin(\pi n) + \frac{1}{\sqrt{2}\pi n} \sin(\frac{\pi n}{2}) - \frac{\sqrt{2}}{\pi^2 n^2} \cos(\pi n x)\Big|_{\frac{1}{2}}^1 \\
&= \frac{\sqrt{2}}{\pi^2 n^2} (2\cos(\frac{\pi n}{2}) - 1 - (-1)^n)
\end{aligned}$$

Thus for n odd $\langle f(x), \phi_n \rangle = 0$. For $n = 2j$ even ($j \in \mathbf{N}$)

$$\langle f(x), \phi_{2j} \rangle = \frac{\sqrt{2}}{4\pi^2 j^2} (2\cos(\pi j) - 2).$$

For j even $\langle f(x), \phi_{2j} \rangle = 0$. For $j = 2k+1$ odd ($k \in \mathbf{N}_0$)

$$\langle f(x), \phi_{4k+2} \rangle = -\frac{\sqrt{2}}{\pi^2 (2k+1)^2}.$$

Thus

$$f(x) = \frac{1}{4} - \sum_{k=0}^{\infty} \frac{2}{\pi^2(2k+1)^2} \cos(2\pi(2k+1)x).$$

We have $f(0) = 0$,

$$f(0) = 0 = \frac{1}{4} - \sum_{k=0}^{\infty} \frac{2}{\pi^2(2k+1)^2}$$

from which follows

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

4. To calculate the entries of the matrix $\tilde{A}$ we calculate the scalar products $\mathbf{a}^T A \mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in \{\Phi^+, \Phi^-, \Psi^+, \Psi^-\}$. We use the indexing scheme $\Phi^+ \to 1$, $\Phi^- \to 2$, $\Psi^+ \to 3$ and $\Psi^- \to 4$ to identify entries in the matrix $\tilde{A}$. Since A is symmetric and real, and the Bell basis is real we have

$$\mathbf{a}^T A \mathbf{b} = \mathbf{b}^T A \mathbf{a}$$

for $\mathbf{a}$ and $\mathbf{b}$ in the Bell basis, i.e. $\tilde{A}$ is symmetric. Let

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_5 & a_6 & a_7 \\ a_3 & a_6 & a_8 & a_9 \\ a_4 & a_7 & a_9 & a_{10} \end{pmatrix}.$$

Since $\tilde{A}$ must be diagonal, we have

$$\begin{aligned}
{\Phi^+}^T A \Phi^- &= \frac{1}{2}(a_1 - a_{10}) = 0 \\
{\Phi^+}^T A \Psi^+ &= \frac{1}{2}(a_2 + a_3 + a_7 + a_9) = 0 \\
{\Phi^+}^T A \Psi^- &= \frac{1}{2}(a_2 - a_3 + a_7 - a_9) = 0 \\
{\Phi^-}^T A \Psi^+ &= \frac{1}{2}(a_2 + a_3 - a_7 - a_9) = 0 \\
{\Phi^-}^T A \Psi^- &= \frac{1}{2}(a_2 - a_3 - a_7 + a_9) = 0 \\
{\Psi^+}^T A \Psi^- &= \frac{1}{2}(a_5 - a_8) = 0
\end{aligned}$$

from which we conclude that

$$a_1 = a_{10}, \quad a_5 = a_8, \quad a_2 = a_3 = a_7 = a_9 = 0.$$

Thus A has the form

$$A = \begin{pmatrix} a_1 & 0 & 0 & a_4 \\ 0 & a_5 & a_6 & 0 \\ 0 & a_6 & a_5 & 0 \\ a_4 & 0 & 0 & a_1 \end{pmatrix}.$$