TEST II (TOETS II) 3B

1. The Fourier expansion is given by

$$f(x) = \sum_{k\in\mathbf{Z}} \langle f(x), \phi_k(x)\rangle \phi_k(x)$$

where $\langle\cdot,\cdot\rangle$ is the scalar product in $L_2(-\pi,\pi)$.
We have $\langle f(x), \phi_0(x)\rangle = 0$. For $k \neq 0$

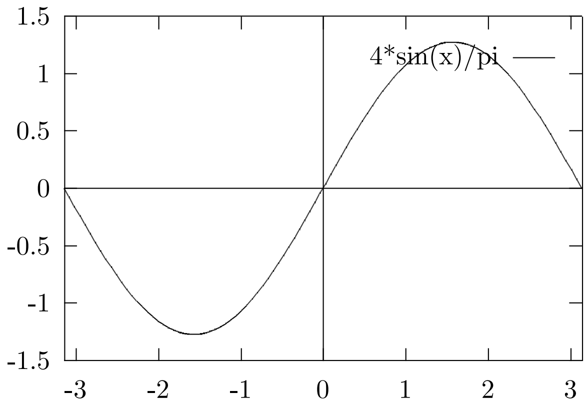
$$\begin{aligned}
\langle f(x), \phi_k(x)\rangle &= \lim_{b\to 0}\int_b^\pi \overline{\phi_k(x)}dx - \lim_{b\to 0}\int_{-\pi}^b \overline{\phi_k(x)}dx \\
&= \lim_{b\to 0}\left(\frac{i}{k\sqrt{2\pi}} \exp(-ikx)\Big|_b^\pi - \frac{i}{k\sqrt{2\pi}} \exp(-ikx)\Big|_{-\pi}^b \right) \\
&= \frac{i}{k\sqrt{2\pi}} \lim_{b\to 0}\left(2(-1)^k - 2\exp(ikb)\right) \\
&= -\frac{2i}{k\sqrt{2\pi}}(1-(-1)^k)
\end{aligned}$$

Thus

$$f(x) = -\sum_{k\in\mathbf{Z}} \frac{2i}{(2k+1)\pi} \exp\left(i(2k+1)x\right).$$

For the approximative solution

$$\begin{aligned}
f(x) \approx a_0\phi_0(x) + a_1\phi_1(x) + a_{-1}\phi_{-1}(x) &= -\frac{2i}{\pi}\exp(ix) + \frac{2i}{\pi}\exp(-ix) \\
&= \frac{4}{\pi}\sin(x)
\end{aligned}$$



2. (a)

$$\begin{aligned}
f(a,b) &= \int_0^\pi (\sin(x) - (ax^2+bx))^2 dx \\
&= \int_0^\pi \left(\sin^2(x) - (2ax^2+2bx)\sin(x) + a^2x^4 + 2abx^3 + b^2x^2\right) dx \\
\int_0^\pi \sin^2(x)dx &= \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = \frac{\pi}{2} \\
\int_0^\pi x\sin(x)dx &= \pi + \int_0^\pi \cos(x)dx = \pi \\
\int_0^\pi x^2\sin(x)dx &= \pi^2 + \int_0^\pi 2x\cos(x)dx \\
&= \pi^2 - \int_0^\pi 2\sin(x)dx = \pi^2 - 4 \\
f(a,b) &= \frac{\pi}{2} - 2a(\pi^2-4) - 2b\pi + \frac{a^2\pi^5}{5} + \frac{ab\pi^4}{2} + \frac{b^2\pi^3}{3}
\end{aligned}$$

(b)

$$\frac{\partial f}{\partial a} = -2\pi^2 + 8 + \frac{2a\pi^5}{5} + \frac{b\pi^4}{2} = 0 \qquad (1)$$

$$\frac{\partial f}{\partial b} = -2\pi + \frac{a\pi^4}{2} + \frac{2b\pi^3}{3} = 0 \qquad (2)$$

$$\frac{\partial^2 f}{\partial a \partial b} = \frac{\pi^4}{2}, \quad \frac{\partial^2 f}{\partial a^2} = \frac{2\pi^5}{5}, \quad \frac{\partial^2 f}{\partial b^2} = \frac{2\pi^3}{3},$$

From (2)

$$a = 4\pi^{-3} - \frac{4b}{3}\pi^{-1}$$

into (1)

$$\begin{aligned}
b &= -\frac{12}{\pi^2} + \frac{240}{\pi^4} \\
a &= +\frac{20}{\pi^3} - \frac{320}{\pi^5}
\end{aligned}$$

The trace and determinant of the Hessian matrix are positive, thus the Hessian matrix is positive definite and the solution is a minimum.

3.

$$H_{mn} = 2^{-\frac{m}{2}} H(2^{-m}x - n) = \begin{cases} 2^{-\frac{m}{2}} & 0 \leq 2^{-m}x - n \leq \frac{1}{2} \\ -2^{-\frac{m}{2}} & \frac{1}{2} \leq 2^{-m}x - n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 2^{-\frac{m}{2}} & 2^m n \leq x \leq 2^m(n + \frac{1}{2}) \\ -2^{-\frac{m}{2}} & 2^m(n + \frac{1}{2}) \leq x \leq 2^m(n+1) \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$H_{11} = \begin{cases} \frac{1}{\sqrt{2}} & 2 \leq x \leq 3 \\ -\frac{1}{\sqrt{2}} & 3 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \quad H_{12} = \begin{cases} \frac{1}{\sqrt{2}} & 4 \leq x \leq 5 \\ -\frac{1}{\sqrt{2}} & 5 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$
$$H_{21} = \begin{cases} \frac{1}{2} & 4 \leq x \leq 6 \\ -\frac{1}{2} & 6 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases} \quad H_{22} = \begin{cases} \frac{1}{2} & 8 \leq x \leq 10 \\ -\frac{1}{2} & 10 \leq x \leq 12 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle H_{mn}(x), H_{kl}(x) \rangle = \int_{-\infty}^{\infty} H_{mn}(x) H_{kl} dx := I_{mnkl}.$$

The non-zero intervals are

$$I_{mn} := (2^m n, 2^m(n+1)), \quad I_{kl} := (2^k l, 2^k(l+1)).$$

We consider the different cases

(a) $m = k$, $n = l$
$$I_{mnkl} = \int_{2^m n}^{2^m(n+1)} 2^{-m} dx = 1.$$

(b) $m = k$, $n \neq l$ $I_{mnkl} = 0$ since $I_{mn} \cap I_{kl} = \emptyset$.

(c) $m \neq k$, suppose without loss of generality that $m < k$. Either $I_{mn} \cap I_{kl} = \emptyset$ ($I_{mnkl} = 0$), or $I_{mn} \subset I_{kl}$ (as shown below). We have the following

$$\begin{aligned} 2^k l \leq 2^m n < 2^k(l + \frac{1}{2}) &\Rightarrow 2^{k-m} l \leq n < 2^{k-m}(l + \frac{1}{2}) \\ &\Rightarrow 2^{k-m} l \leq n + 1 \leq 2^{k-m}(l + \frac{1}{2}) \\ &\Rightarrow 2^k l \leq 2^m(n+1) \leq 2^k(l + \frac{1}{2}) \end{aligned}$$

$$
\begin{aligned}
2^k(l+\frac{1}{2}) \le 2^m n < 2^k(l+1) &\Rightarrow 2^k(l+\frac{1}{2}) \le 2^m(n+1) \le 2^k(l+1) \\
2^k l < 2^m(n+1) \le 2^k(l+\frac{1}{2}) &\Rightarrow 2^k l \le 2^m n \le 2^k(l+\frac{1}{2}) \\
2^k(l+\frac{1}{2}) < 2^m(n+1) \le 2^k(l+1) &\Rightarrow 2^k(l+\frac{1}{2}) \le 2^m n \le 2^k(l+1)
\end{aligned}
$$

which gives

$$
I_{mnkl} = \pm \int_{2^m n}^{2^m(n+\frac{1}{2})} 2^{-\frac{1}{2}(m+k)} dx \mp \int_{2^m(n+\frac{1}{2})}^{2^m(n+1)} 2^{-\frac{1}{2}(m+k)} dx = 0
$$

Thus $I_{mnkl} = \delta_{mk}\delta_{nl}$.

$$
\begin{aligned}
\langle f(x), H_{mn}(x)\rangle &= \int_{-\infty}^{\infty} f(x) H_{mn}(x) dx \\
&= \int_{2^m n}^{2^m(n+\frac{1}{2})} 2^{-\frac{m}{2}} e^{-|x|} dx - \int_{2^m(n+\frac{1}{2})}^{2^m(n+1)} 2^{-\frac{m}{2}} e^{-|x|} dx \\
&= 2^{-\frac{m}{2}} \begin{cases} -e^{-x}\Big|_{2^m n}^{2^m(n+\frac{1}{2})} + e^{-x}\Big|_{2^m(n+\frac{1}{2})}^{2^m(n+1)} & n \ge 0 \\ e^{x}\Big|_{2^m n}^{2^m(n+\frac{1}{2})} - e^{x}\Big|_{2^m(n+\frac{1}{2})}^{2^m(n+1)} & n < 0 \end{cases} \\
&= -\frac{n - \delta_{n,0}}{|n| + \delta_{n,0}} 2^{-\frac{m}{2}} (2e^{-2^m|n+\frac{1}{2}|} - e^{-2^m|n|} - e^{-2^m|n+1|}).
\end{aligned}
$$

The expansion is given by

$$
f(x) = \sum_{m,n\in\mathbf{Z}} \langle f(x), H_{mn}(x)\rangle H_{mn}(x).
$$