TEST I (TOETS I) 3B

1.

$$
\begin{aligned}
||\cos(x)||^2 &= \langle \cos(x), \cos(x) \rangle \\
&= \int_{-\pi}^{\pi} \cos(x)\overline{\cos(x)}dx \\
&= \int_{-\pi}^{\pi} \cos^2(x)dx \\
&= \int_{-\pi}^{\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right) dx \\
&= \left[\frac{x}{2} + \frac{1}{4}\sin(2x)\right]_{-\pi}^{\pi} \\
&= \pi
\end{aligned}
$$

Thus $||\cos(x)|| = \sqrt{\pi} < \infty$.

Consider $f(x) = x$ and $g(x) = 1$.

$$
\begin{aligned}
\langle f(x), \cos(x) \rangle &= \int_{-\pi}^{\pi} x\cos(x)dx \\
&= x\sin(x)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin(x)dx \\
&= 0 \\
\langle g(x), \cos(x) \rangle &= \int_{-\pi}^{\pi} \cos(x)dx \\
&= 0 \\
\langle g(x), f(x) \rangle &= \int_{-\pi}^{\pi} xdx \\
&= 0
\end{aligned}
$$

2. p cannot exceed 4 since that would imply $\dim(\mathbf{R}^4) > 4$.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

$$\begin{aligned}
\sum_{i=1}^{4} \mathbf{x}_i &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} \\
&+ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.
\end{aligned}$$

The eigenvalue is 4 with multiplicity 4. The eigenvectors are all $\mathbf{x} \in \mathbf{R}^4$.

3. Without loss of generality we can assume that $p > q$. First we must prove for $q \geq 4$ the inequality

$$\sum_{n=q}^{p-1} \frac{1}{n!} < \frac{p-q}{pq}.$$

We do this by induction over p

- For $p = q+1$ we have $\frac{1}{q!} < \frac{1}{(q+1)q}$ for $q \geq 4$.
- Assume that $(p = q+k)$

$$\sum_{n=q}^{q+k-1} \frac{1}{n!} < \frac{k}{(q+k)q}.$$

It follows that $(p = q+k+1)$

$$\begin{aligned}
\sum_{n=q}^{q+k} \frac{1}{n!} &< \frac{k}{(q+k)q} + \frac{1}{(q+k)!} \\
&= \frac{1}{q(q+k+1)} \left(\frac{k(q+k+1)}{q+k} + \frac{q(q+k+1)}{(q+k)!} \right) \\
&< \frac{1}{q(q+k+1)} \left(k + \frac{k}{q+k} + \frac{q}{q+k} \right) \\
&= \frac{k+1}{q(q+k+1)}
\end{aligned}$$

Now

$$||s_p - s_q|| = \left| \sum_{j=1}^{p} \frac{1}{(j-1)!} - \sum_{j=1}^{q} \frac{1}{(j-1)!} \right| = \left| \sum_{j=q}^{p-1} \frac{1}{j!} \right| < \left| \frac{p-q}{pq} \right|$$

So for $m_\epsilon > \max\{4, \frac{2}{\epsilon}\}$ and $p, q > m_\epsilon$

$$\left| \frac{p-q}{pq} \right| < \frac{p}{pq} + \frac{q}{pq} = \frac{1}{q} + \frac{1}{p} < \frac{1}{m_\epsilon} + \frac{1}{m_\epsilon} < \epsilon.$$

4. (a) Setting an arbitrary linear combination equal to zero

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

leads to the equations

$$\begin{aligned} d &= 0 \\ c + d &= 0 \\ b + c + d &= 0 \\ a + b + c + d &= 0 \end{aligned}$$

Thus $a = b = c = d = 0$.

(b) Setting

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u_4} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

we find the orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 = \mathbf{e}_1 \\ \mathbf{v}_2 &= \mathbf{u}_2 - \mathbf{u}_2^T \mathbf{v}_1 = \mathbf{e}_2 \\ \mathbf{v}_3 &= \mathbf{u}_3 - \mathbf{u}_3^T \mathbf{v}_1 - \mathbf{u}_3^T \mathbf{v}_2 = \mathbf{e}_3 \\ \mathbf{v}_4 &= \mathbf{u}_4 - \mathbf{u}_4^T \mathbf{v}_1 - \mathbf{u}_4^T \mathbf{v}_2 - \mathbf{u}_4^T \mathbf{v}_3 = \mathbf{e}_4 \end{aligned}$$

where

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

is the standard basis.