TEST XI (TOETS XI) 3B

1.

$$U^*_{QFT} = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} e^{i2\pi kj/2^n} |j\rangle\langle k|,$$

$$\begin{aligned} U_{QFT}U^*_{QFT} &= \frac{1}{2^n} \sum_{j,k,l,m=0}^{2^n-1} e^{i2\pi(kj-lm)/2^n} |j\rangle\langle k|l\rangle\langle m| \\ &= \frac{1}{2^n} \sum_{j,k,m=0}^{2^n-1} e^{i2\pi(kj-km)/2^n} |j\rangle\langle m| \end{aligned}$$

We have for $j = m$, $e^{i2\pi(kj-km)/2^n} = 1$. Thus for $j, m = 0, 1, \ldots, 2^n - 1$

$$\begin{aligned} \sum_{k=0}^{2^n-1} (e^{i2\pi(j-m)/2^n})^k &= 2^n, \quad j = m \\ \sum_{k=0}^{2^n-1} (e^{i2\pi(j-m)/2^n})^k &= \frac{1 - e^{i2\pi(j-m)}}{1 - e^{i2\pi(j-m)/2^n}} = 0, \quad j \neq m \end{aligned}$$

Thus

$$U_{QFT}U^*_{QFT} = \sum_{j}^{2^n-1} |j\rangle\langle j| = I.$$

2.

$$U^*_{IA} = \sum_{j=0}^{2^n-1} \sum_{k=0}^{2^n-1} \left(\frac{2}{2^n} - \delta_{jk}\right) |k\rangle\langle j| = U_{IA}$$

$$\begin{aligned} U_{IA}U^*_{IA} = U^2_{IA} &= \sum_{j,k,l,m=0}^{2^n-1} \left(\frac{2}{2^n} - \delta_{jk}\right) \left(\frac{2}{2^n} - \delta_{lm}\right) |k\rangle\langle j|m\rangle\langle l| \\ &= \sum_{j,k,l=0}^{2^n-1} \left(\frac{2}{2^n} - \delta_{jk}\right) \left(\frac{2}{2^n} - \delta_{lj}\right) |k\rangle\langle l| \end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{2^n-1}\left(\frac{2}{2^n}-\delta_{jk}\right)\left(\frac{2}{2^n}-\delta_{lj}\right) &= \sum_{j=0}^{2^n-1}\left(\frac{4}{2^{2n}}-\delta_{jk}\frac{2}{2^n}-\delta_{lj}\frac{2}{2^n}+\delta_{jk}\delta_{lj}\right) \\
&= \frac{4}{2^n}-\frac{2}{2^n}-\frac{2}{2^n}+\sum_{j=0}^{2^n-1}\delta_{jk}\delta_{lk} \\
&= \delta_{lk}\sum_{j=0}^{2^n-1}\delta_{jk} \\
&= \delta_{lk}
\end{aligned}$$

Thus

$$U_{IA}U_{IA}^{*} = \sum_{j}^{2^n-1}|j\rangle\langle j| = I.$$

3. Applying

   $\alpha I\otimes I\otimes I+\beta I\otimes U_{NOT}\otimes U_{NOT}+\delta I\otimes U_P\otimes U_P+\gamma I\otimes(U_P U_{NOT})\otimes(U_P U_{NOT})$

   to the state

   $$\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\otimes|\psi\rangle$$

   yields

   $$\begin{aligned}
   & \alpha\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\otimes|\psi\rangle \\
   + \quad & \beta\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)\otimes(a|1\rangle+b|0\rangle) \\
   + \quad & \delta\frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)\otimes(a|0\rangle-b|1\rangle) \\
   + \quad & \gamma\frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)\otimes(a|1\rangle-b|0\rangle)
   \end{aligned}$$

   Thus we measure the first two qubits in the Bell basis and apply the corresponding transform to the last qubit to obtain $|\psi\rangle$.

Measure	Transform
$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)$	I
$\frac{1}{\sqrt{2}}(\|01\rangle+\|10\rangle)$	U_{NOT}
$\frac{1}{\sqrt{2}}(\|00\rangle-\|11\rangle)$	U_P
$\frac{1}{\sqrt{2}}(\|01\rangle-\|10\rangle)$	$U_{NOT}U_P$