Applied Mathematics 3B
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Semester Test 2

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1. $\sigma_x^2 = I$.

$$e^{-i\sigma_x t/\hbar} = \cosh(-it/\hbar)I + \sinh(-it/\hbar)\sigma_x$$

$$\sigma_x \psi(0) = \psi(0)$$

$$\psi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\frac{t}{\hbar} - i\sin\frac{t}{\hbar} \\ \cos\frac{t}{\hbar} - i\sin\frac{t}{\hbar} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-it/\hbar} \\ e^{-it/\hbar} \end{pmatrix}$$

$$\begin{aligned} \langle \psi(t), \sigma_y \psi(t) \rangle &= \frac{1}{2} \langle (\cos\frac{t}{\hbar} - i\sin\frac{t}{\hbar}, \cos\frac{t}{\hbar} - i\sin\frac{t}{\hbar})^T, \\ & \quad i(-\cos\frac{t}{\hbar} + i\sin\frac{t}{\hbar}, \cos\frac{t}{\hbar} - i\sin\frac{t}{\hbar})^T \rangle \\ &= \frac{1}{2}\left[i(\cos^2\frac{t}{\hbar} + \sin^2\frac{t}{\hbar}) - i(\cos^2\frac{t}{\hbar} + \sin^2\frac{t}{\hbar}) \right] = 0 \end{aligned}$$

$$\begin{aligned} \sigma_y(t) &= \begin{pmatrix} \cos\frac{t}{\hbar} & i\sin\frac{t}{\hbar} \\ i\sin\frac{t}{\hbar} & \cos\frac{t}{\hbar} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{t}{\hbar} & -i\sin\frac{t}{\hbar} \\ -i\sin\frac{t}{\hbar} & \cos\frac{t}{\hbar} \end{pmatrix} \\ &= \begin{pmatrix} -\sin\frac{t}{\hbar} & -i\cos\frac{t}{\hbar} \\ i\cos\frac{t}{\hbar} & \sin\frac{t}{\hbar} \end{pmatrix} \begin{pmatrix} \cos\frac{t}{\hbar} & -i\sin\frac{t}{\hbar} \\ -i\sin\frac{t}{\hbar} & \cos\frac{t}{\hbar} \end{pmatrix} \\ &= \begin{pmatrix} -2\sin\frac{t}{\hbar}\cos\frac{t}{\hbar} & -i(\cos^2\frac{t}{\hbar} - \sin^2\frac{t}{\hbar}) \\ i(\cos^2\frac{t}{\hbar} - \sin^2\frac{t}{\hbar}) & -2\sin\frac{t}{\hbar}\cos\frac{t}{\hbar} \end{pmatrix} \\ &= \begin{pmatrix} -\sin\frac{2t}{\hbar} & -i\cos\frac{2t}{\hbar} \\ i\cos\frac{2t}{\hbar} & \sin\frac{2t}{\hbar} \end{pmatrix} \end{aligned}$$

$$\langle \psi(0), \sigma_y(t)\psi(0) \rangle = \frac{1}{2} \left\langle (1,1)^T, \left(-\left(i\cos\frac{2t}{\hbar} + \sin\frac{2t}{\hbar} \right), \left(i\cos\frac{2t}{\hbar} + \sin\frac{2t}{\hbar} \right) \right)^T \right\rangle = 0$$

2. We use the fact that (expansion in the Bell basis)

$$\begin{aligned} |00\rangle &= \frac{1}{2}(|00\rangle + |11\rangle) + \frac{1}{2}(|00\rangle - |11\rangle) \\ |11\rangle &= \frac{1}{2}(|00\rangle + |11\rangle) - \frac{1}{2}(|00\rangle - |11\rangle) \end{aligned}$$

$$
\begin{aligned}
|01\rangle &= \frac{1}{2}(|01\rangle + |10\rangle) + \frac{1}{2}(|01\rangle - |10\rangle) \\
|10\rangle &= \frac{1}{2}(|01\rangle + |10\rangle) - \frac{1}{2}(|01\rangle - |10\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) &= \frac{1}{\sqrt{2}}(a|000\rangle - a|011\rangle + b|100\rangle - b|111\rangle) \\
&= \frac{1}{2\sqrt{2}}a((|00\rangle + |11\rangle) + (|00\rangle - |11\rangle)) \otimes |0\rangle \\
&- \frac{1}{2\sqrt{2}}a((|01\rangle + |10\rangle) + (|01\rangle - |10\rangle)) \otimes |1\rangle \\
&+ \frac{1}{2\sqrt{2}}b((|01\rangle + |10\rangle) - (|01\rangle - |10\rangle)) \otimes |0\rangle \\
&- \frac{1}{2\sqrt{2}}b((|00\rangle + |11\rangle) - (|00\rangle - |11\rangle)) \otimes |1\rangle \\
&= \frac{1}{2\sqrt{2}}(|00\rangle + |11\rangle) \otimes (a|0\rangle - b|1\rangle) \\
&+ \frac{1}{2\sqrt{2}}(|00\rangle - |11\rangle) \otimes (a|0\rangle + b|1\rangle) \\
&- \frac{1}{2\sqrt{2}}(|01\rangle + |10\rangle) \otimes (a|1\rangle - b|0\rangle) \\
&- \frac{1}{2\sqrt{2}}(|01\rangle - |10\rangle) \otimes (a|1\rangle + b|0\rangle)
\end{aligned}
$$

Thus we measure the first two qubits in the Bell basis and apply the following transforms to the last qubit

Measure	Transform
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	$\|0\rangle\langle 0\| - \|1\rangle\langle 1\|$
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	I
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	$\|1\rangle\langle 0\| - \|0\rangle\langle 1\|$
$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	$-U_{NOT}$

3. We note that we work in a two dimensional Hilbert space and $|\{|0_H\rangle, |1_H\rangle\}| = 2$. Also

$$
a|0\rangle + b|1\rangle = \frac{1}{\sqrt{2}}(a+b)|0_H\rangle + \frac{1}{\sqrt{2}}(a-b)|1_H\rangle
$$

$$a|0_H\rangle + b|1_H\rangle \;=\; \frac{1}{\sqrt{2}}(a+b)|0\rangle + \frac{1}{\sqrt{2}}(a-b)|1\rangle.$$

Thus for $a|0_H\rangle + b|1_H\rangle = 0$, $a = b = 0$. We find

$$\begin{aligned}
\langle 0_H|0_H\rangle &= \frac{1}{2}(\langle 0|0\rangle + \langle 0|1\rangle + \langle 1|0\rangle + \langle 1|1\rangle) = 1 \\
\langle 1_H|1_H\rangle &= \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle - \langle 1|0\rangle + \langle 1|1\rangle) = 1 \\
\langle 0_H|1_H\rangle &= \frac{1}{2}(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle) = 0
\end{aligned}$$

a) $|\langle 0|0_h\rangle|^2 = \frac{1}{2}|(\langle 0|0\rangle + \langle 0|1\rangle|^2 = \frac{1}{2}$
b) $|\langle 0|1_h\rangle|^2 = \frac{1}{2}|(\langle 0|0\rangle - \langle 0|1\rangle|^2 = \frac{1}{2}$
Thus measurement projects the state $|0\rangle$ onto $|0_H\rangle$ and $|1_H\rangle$ with equal probability. Starting with $|0\rangle$, we can obtain $|0_H\rangle$ and $|1_H\rangle$ by measurement and applying U_PS as follows

Desired state	Measure	Transform
$\|0_H\rangle$	$\|0_H\rangle$	I
$\|0_H\rangle$	$\|1_H\rangle$	U_{PS}
$\|1_H\rangle$	$\|0_H\rangle$	U_{PS}
$\|1_H\rangle$	$\|1_H\rangle$	I

c)

$$\begin{aligned}
|0_H\rangle \otimes |0_H\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \\
U_f|0_H\rangle \otimes |0_H\rangle &= \frac{1}{2}(|0f(0)\rangle + |0\overline{f(0)}\rangle + |1f(1)\rangle + |1\overline{f(1)}\rangle) \\
&= \frac{1}{2}((1-f(0))|00\rangle + f(0)|01\rangle + f(0)|00\rangle + (1-f(0))|01\rangle \\
&\quad +(1-f(1))|10\rangle + f(1)|11\rangle + f(1)|10\rangle + (1-f(1))|11\rangle) \\
&= |0_H\rangle \otimes |0_H\rangle
\end{aligned}$$

d)

$$\begin{aligned}
|0_H\rangle \otimes |1_H\rangle &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \\
U_f|0_H\rangle \otimes |1_H\rangle &= \frac{1}{2}(|0f(0)\rangle - |0\overline{f(0)}\rangle + |1f(1)\rangle - |1\overline{f(1)}\rangle)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}((1-f(0))|00\rangle + f(0)|01\rangle - f(0)|00\rangle - (1-f(0))|01\rangle \\
&+ (1-f(1))|10\rangle + f(1)|11\rangle - f(1)|10\rangle - (1-f(1))|11\rangle) \\
&= \frac{1}{2}((1-2f(0))|00\rangle - (1-2f(0))|01\rangle \\
&+ (1-2f(1))|10\rangle - (1-2f(1))|11\rangle) \\
&= \frac{1}{2}\left((-1)^{f(0)}|00\rangle - (-1)^{f(0)}|01\rangle - (-1)^{f(1)}|10\rangle - (-1)^{f(1)}|11\rangle\right) \\
&= \frac{1}{2}\left((-1)^{f(0)}|0\rangle \otimes (|0\rangle - |1\rangle) + (-1)^{f(1)}|1\rangle(|0\rangle - |1\rangle)\right) \\
&= \frac{1}{\sqrt{2}}(-1)^{f(0)}\left((|0\rangle - (-1)^{f(0)+f(1)}|1\rangle) \otimes |1_H\rangle\right) \\
&= \frac{1}{\sqrt{2}}(-1)^{f(0)}\left((|0\rangle - (-1)^{f(0)\oplus f(1)}|1\rangle) \otimes |1_H\rangle\right) \\
&= (-1)^{f(0)}|f(0) \oplus f(1)\rangle \otimes |1_H\rangle
\end{aligned}
$$

Note that $f(0) \oplus f(1)$ is 0 when f is constant, and 1 when f is balanced. Thus we have solved Deutsch's problem.

4. a)

$$
\begin{aligned}
\sin\theta\cos\theta &= \sin\phi\cos\phi \\
\sin 2\theta &= \sin 2\phi \\
\theta = \phi + k\pi \quad or \quad \theta &= \frac{\pi}{2} - \phi + k\pi, \quad k \in \mathbf{Z}
\end{aligned}
$$

b)

$$
\cos\theta = \sin\theta \rightarrow \theta = \frac{\pi}{4} + k\pi, \quad k \in \mathbf{Z}
$$

c)

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}(U_H \otimes U_H)\left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}\right) \\
&= \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}1\\-1\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}(1,0,0,1)^T
\end{aligned}
$$

which is an entangled state $(1 \neq 0)$.

d)

$$
\frac{1}{\sqrt{2}}U_{CNOT}(1,0,0,1)^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}
$$

which is not entangled.