Applied Mathematics 3B
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Semester Test 1

Semester Toets 1

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1. $k \neq l$

$$
\begin{aligned}
\langle \phi_k(x), \phi_k(x) \rangle &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i k x} e^{-2\pi i k x} dx = [x]_{-\frac{1}{2}}^{\frac{1}{2}} = 1. \\
\langle \phi_k(x), \phi_l(x) \rangle &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i k x} e^{-2\pi i l x} dx \\
&= \left[\frac{1}{2\pi i(k-l)} e^{2\pi i (k-l) x} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2\pi i(k-l)}(1-1) = 0. \\
\langle \psi_k(x), \psi_k(x) \rangle &= \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\sin^2(2\pi k x) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - \cos(4\pi k x)) dx \\
&= \left[x - \frac{1}{4\pi k} \cos(4\pi k x) \right]_{-\frac{1}{2}}^{\frac{1}{2}} = 1 \\
\langle \psi_k(x), \phi_l(x) \rangle &= \int_{-\frac{1}{2}}^{\frac{1}{2}} 2\sin(2\pi k x)\sin(2\pi l x) dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} (\cos(2\pi(k-l)x) - \cos(2\pi(k+l)x)) dx \\
&= \left[\frac{1}{2\pi(k-l)} \sin(2\pi(k-l)x) - \frac{1}{2\pi(k+l)} \sin(2\pi(k+l)x) \right]_{-\frac{1}{2}}^{\frac{1}{2}} = 0
\end{aligned}
$$

$$
\begin{aligned}
\langle f(x), \phi_0(x) \rangle &= -\int_{-\frac{1}{2}}^{0} dx + \int_{0}^{\frac{1}{2}} dx = 0 \\
k \neq 0 \quad \langle f(x), \phi_k(x) \rangle &= -\int_{-\frac{1}{2}}^{0} e^{-2\pi i k x} dx + \int_{0}^{\frac{1}{2}} e^{-2\pi i k x} dx = 0 \\
&= \frac{1}{2\pi i k x} \left(\left[e^{-2\pi i k x} \right]_{-\frac{1}{2}}^{0} + \left[e^{-2\pi i k x} \right]_{0}^{\frac{1}{2}} \right) \\
&= \frac{1}{\pi i k}(1 - (-1)^k)
\end{aligned}
$$

$$
\begin{aligned}
\langle f(x), \psi_k(x)\rangle &= -\int_{-\frac{1}{2}}^{0} \sqrt{2}\sin(2\pi kx)dx + \int_0^{\frac{1}{2}} \sqrt{2}\sin(2\pi kx)dx \\
&= +\left[\frac{1}{\sqrt{2}\pi k}\cos(2\pi kx)\right]_{-\frac{1}{2}}^{0} - \left[\frac{1}{\sqrt{2}\pi k}\cos(2\pi kx)\right]_0^{\frac{1}{2}} \\
&= \frac{2}{\sqrt{2}\pi k}(1-(-1)^k)
\end{aligned}
$$

Thus

$$
\begin{aligned}
f(x) &= \sum_{k\in\mathbf{Z},k\neq 0} \frac{1}{\pi ik}(1-(-1)^k)e^{2\pi ikx} \\
&= \sum_{k\in\mathbf{Z},k\neq 0} \frac{1}{\pi ik}(1-(-1)^k)(\cos(2\pi kx) + i\sin(2\pi kx)) \\
&= \sum_{k=1,k\ odd}^{\infty} \frac{4}{\pi k}\sin(2\pi kx) \\
f(x) &= \sum_{k\in\mathbf{N}} \frac{2}{\sqrt{2}\pi k}(1-(-1)^k)\sqrt{2}\sin(2\pi kx) \\
&= \sum_{k=1,k\ odd}^{\infty} \frac{4}{\pi k}\sin(2\pi kx)
\end{aligned}
$$

2. Let

$$
A := \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.
$$

$$
A^2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix} = 2A.
$$

Thus we have

$$
A^{2n} = 2^{n-1}A^2, \qquad A^{2n+1} = 2^n A
$$

and

$$
\begin{aligned}
\exp(-i\hat{S}_x t/\hbar) &= I + \sum_{j=0}^{\infty} \frac{(-it/\sqrt{2})^{2j+1}}{(2j+1)!}2^j A + \sum_{j=1}^{\infty} \frac{(-it/\sqrt{2})^{2j}}{(2j)!}2^{j-1}A^2 \\
&= I + \frac{1}{\sqrt{2}}\sum_{j=0}^{\infty} \frac{(-it)^{2j+1}}{(2j+1)!}A + \frac{1}{2}\sum_{j=1}^{\infty} \frac{(-it)^{2j}}{(2j)!}A^2
\end{aligned}
$$

$$
\begin{aligned}
&= I + \frac{1}{\sqrt{2}}\sinh(-it)A + \frac{1}{2}(\cosh(-it)-1)A^2 \\
&= I - \frac{i}{\sqrt{2}}\sin(t)A + \frac{1}{2}(\cos(t)-1)A^2 \\
&= \begin{pmatrix} \frac{1}{2}\cos(t)+\frac{1}{2} & -\frac{1}{\sqrt{2}}\sin(t) & \frac{1}{2}-\frac{1}{2}\cos(t) \\ \frac{1}{\sqrt{2}}\sin(t) & \cos(t) & -\frac{1}{\sqrt{2}}\sin(t) \\ \frac{1}{2}-\frac{1}{2}\cos(t) & \frac{1}{\sqrt{2}}\sin(t) & \frac{1}{2}\cos(t)+\frac{1}{2} \end{pmatrix}.
\end{aligned}
$$

$$
\psi(t) = \exp(-i\hat{S}_x t/\hbar)\psi(0) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1-\frac{1}{\sqrt{2}}\sin(t) \\ \cos(t) \\ 1+\frac{1}{\sqrt{2}}\sin(t) \end{pmatrix}.
$$

The probability of finding $\psi(t)$ in the initial state is

$$
\begin{aligned}
|\langle\psi(t)|\psi(0)\rangle|^2 &= \frac{1}{9}|2+\cos(t)|^2 \\
&= \frac{1}{9}(2+\cos(t))^2
\end{aligned}
$$

3. The eigenvalue of I is $\lambda_{1,0} = \lambda_{1,1} = 1$ with multiplicity 2. The eigenspace corresponding to this eigenvalue is $\mathbf{C}^2$. In other words the normalized eigenstates are $(1,0)^T$ and $(0,1)^T$ (we can choose any orthonormal basis). The matrices σ_x, σ_y and σ_z are Hermitian, unitary and have zero trace. Thus the eigenvalues are $\lambda_{x,0} = \lambda_{y,0} = \lambda_{z,0} = -1$ and $\lambda_{x,1} = \lambda_{y,1} = \lambda_{z,1} = 1$. The corresponding normalized eigenvectors are

$\lambda_{1,0}$	$(1,0)^T$
$\lambda_{1,1}$	$(0,1)^T$
$\lambda_{x,0}$	$\frac{1}{\sqrt{2}}(1,-1)^T$
$\lambda_{x,1}$	$\frac{1}{\sqrt{2}}(1,1)^T$
$\lambda_{y,0}$	$\frac{1}{\sqrt{2}}(1,i)^T$
$\lambda_{y,1}$	$\frac{1}{\sqrt{2}}(1,-i)^T$
$\lambda_{z,0}$	$(0,1)^T$
$\lambda_{z,1}$	$(1,0)^T$

For the probabilities we take the magnitude squared of the inner product between the eigenvector and the state. For the observable I both eigenvalues are 1 and we sum the probabilities for the corresponding eigenvectors.

$\lambda_{1,0}$	1
$\lambda_{1,1}$	1
$\lambda_{x,0}$	1
$\lambda_{x,1}$	0
$\lambda_{y,0}$	$\frac{1}{2}$
$\lambda_{y,1}$	$\frac{1}{2}$
$\lambda_{z,0}$	$\frac{1}{2}$
$\lambda_{z,1}$	$\frac{1}{2}$

4. An arbitrary state in the Hilbert space can be written as

$$|\psi\rangle := a|0\rangle + b|1\rangle$$

where $a, b \in \mathbf{C}$.

$$\begin{aligned} U_H U_H |\psi\rangle &= U_H \frac{1}{\sqrt{2}} (a|0\rangle + a|1\rangle + b|0\rangle - b|1\rangle) \\ &= \frac{1}{2} (2a|0\rangle + 2b|1\rangle) \\ &= (a|0\rangle + b|1\rangle) \end{aligned}$$

Thus $U_H U_H = I$.

$$\begin{aligned} (I \otimes U_H) U_{PS(\pi)} (I \otimes U_H)|ab\rangle &= (I \otimes U_H) U_{PS(\pi)} \frac{1}{\sqrt{2}} |a\rangle \otimes (|0\rangle + (-1)^b |1\rangle) \\ &= (I \otimes U_H) \frac{1}{\sqrt{2}} |a\rangle \otimes (|0\rangle + (-1)^{a+b} |1\rangle \\ &= \frac{1}{2} |a, a \oplus b\rangle \\ (I \otimes U_H) U_{CNOT} (I \otimes U_H)|ab\rangle &= (I \otimes U_H) U_{CNOT} \frac{1}{\sqrt{2}} |a\rangle \otimes (|0\rangle + (-1)^b |1\rangle \\ &= \begin{cases} (I \otimes U_H) \frac{1}{\sqrt{2}} |a\rangle \otimes (|0\rangle + (-1)^b |1\rangle & a = 0 \\ (I \otimes U_H) \frac{1}{\sqrt{2}} |a\rangle \otimes (|1\rangle + (-1)^b |0\rangle & a = 1 \end{cases} \\ &= (I \otimes U_H) \frac{1}{\sqrt{2}} |a\rangle \otimes (-1)^{a \cdot b} (|0\rangle + (-1)^b |1\rangle \\ &= (-1)^{a \cdot b} |ab\rangle \end{aligned}$$

The first computation is U_{CNOT}, the second is $U_{PS(\pi)}$.