TWK2A
Solutions by substitutions
Solutions

1. Letting $x = vy$ we have

$$\begin{aligned}
y(vdy + ydv) - 2(vy + y)dy &= 0 \\
ydv - (v+2)dy &= 0 \\
\Rightarrow \frac{dv}{v+2} - \frac{dy}{y} &= 0 \\
\Rightarrow \ln|v+2| - \ln|y| &= c \\
\Rightarrow \ln\left|\frac{x}{y} + 2\right| - \ln|y| &= c \\
\Rightarrow x + 2y &= cy^2
\end{aligned}$$

2. Letting $y = ux$ we have

$$\begin{aligned}
(x + 3ux)dx - (3x + ux)(udx + xdu) &= 0 \\
\Rightarrow (u^2 - 1)dx + x(u+3)du &= 0 \\
\Rightarrow \frac{dx}{x} + \frac{u+3}{(u-1)(u+1)}du &= 0 \\
\Rightarrow \ln|x| + 2\ln|u-1| - \ln|u+1| &= c \\
\Rightarrow \frac{x(u-1)^2}{u+1} &= c \\
\Rightarrow x\left(\frac{y}{x} - 1\right)^2 &= c\left(\frac{y}{x} + 1\right) \\
\Rightarrow (y-x)^2 &= c(y+x)
\end{aligned}$$

3. Letting $y = ux$ we have

$$\begin{aligned}
(x + uxe^u)dx - xe^u(udx + xdu) &= 0 \\
\Rightarrow dx - xe^u du &= 0 \\
\Rightarrow \frac{dx}{x} - e^u du &= 0 \\
\Rightarrow \ln|x| - e^u &= c \\
\Rightarrow \ln|x| - e^{y/x} &= c.
\end{aligned}$$

4. From $y' - y = e^x y^2$ and $w = y^{-1}$ we obtain

$$\frac{dw}{dx} + w = -e^x.$$

An integrating factor is e^x so that

$$e^x w = -\frac{1}{2}e^{2x} + c \Rightarrow y^{-1} = -\frac{1}{2}e^x + ce^{-x}.$$

5. From $y' + y = y^{-1/2}$ and $w = y^{\frac{3}{2}}$ we obtain

$$\frac{dw}{dx} + \frac{3}{2}w = \frac{3}{2}.$$

An integrating factor is $e^{3x/2}$ so that

$$e^{\frac{3x}{2}}w = 3^{\frac{3x}{2}} + c \Rightarrow y^{\frac{3}{2}} = 1 + ce^{-\frac{3x}{2}}.$$

If $y(0) = 4$ then $c = 7$ and

$$y^{\frac{3}{2}} = 1 + 7e^{-\frac{3x}{2}}.$$

6. With $u = y - x + 5$ we have

$$\frac{du}{dx} = \frac{dy}{dx} - 1.$$

Hence,

$$\frac{du}{dx} + 1 = 1 + e^u \Rightarrow \ e^{-u}du = dx.$$

Thus, $-e^{-u} = x + c$ and so

$$-e^{-(y-x+5)} = x + c \Rightarrow y\left(x\right) = x - 5 - \ln\left(-x - c\right).$$

Note that for this solution the interval of definition must be such that $-x - c > 0 \Rightarrow x < -c$. In other words, the interval of definition of the solution here is influenced by the value of the integration constant.

7. The substitutions $y = y_1 + u$ and

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

lead to

$$\begin{aligned}\frac{dy_1}{dx} + \frac{du}{dx} &= P + Q(y_1 + u) + R(y_1 + u)^2 \\ &= P + Qy_1 + Ry_1^2 + Qu + 2y_1Ru + Ru^2\end{aligned}$$

or

$$\frac{du}{dx} - (Q + 2y_1R)u = Ru^2.$$

This is a Bernoulli equation with $n = 2$ which can be reduced to the linear equation

$$\frac{dw}{dx} + (Q + 2y_1R)w = -R$$

by the substitution $w = u^{-1}$. For the given DE, identify $P(x) = -4/x^2$, $Q(x) = -1/x$, and $R(x) = 1$. Then $\frac{dw}{dx} + \left(-\frac{1}{x} + \frac{4}{x}\right) w = -1$. An integrating factor is x^3 so that $x^3 w = -\frac{1}{4}x^4 + c$ or $u = \left[-\frac{1}{4}x + cx^3\right]^{-1}$. Thus,

$$y\left(x\right) = \frac{2}{x} + \left[-\frac{1}{4}x + cx^3\right]^{-1}.$$

8.

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2} \qquad (1)$$

Rearranging:

$$\left(y + \sqrt{x^2 + y^2}\right) dx - x dy = 0$$

For

$$M(x, y) \equiv y + \sqrt{x^2 + y^2} \quad \text{and} \quad N(x, y) \equiv -x$$

we have

$$\begin{aligned}M(tx, ty) &= ty + \sqrt{t^2x^2 + t^2y^2} = t\left(y + \sqrt{x^2 + y^2}\right) = tM(x, y) \\ N(tx, ty) &= t(-x) = tN(x, y).\end{aligned}$$

Thus, (1) is homogeneous and we use the substitution

$$u = \frac{y}{x} \tag{2}$$

which implies

$$\begin{array}{l} y = ux \\ \frac{dy}{dx} = y + x\frac{du}{dx} \end{array} . \tag{3}$$

Substitute (3) into (1):

$$\begin{aligned} x\left(u + x\frac{du}{dx}\right) - ux &= \sqrt{x^2 + x^2u^2} \\ x\frac{du}{dx} &= \sqrt{1 + u^2} \end{aligned}$$

Separate variables and integrate:

$$\begin{aligned} \int \frac{du}{\sqrt{1+u^2}} &= \int \frac{dx}{x} + c_1 \\ \Rightarrow \ln\left|u + \sqrt{1+u^2}\right| &= \ln|x| + c_1 \\ \Rightarrow \ln\left|\frac{u + \sqrt{1+u^2}}{x}\right| &= c_1 \end{aligned} \tag{4}$$

The integral on the LHS of (4) may be obtained using the substitution $u = \tan\theta$. Now use (2):

$$\begin{aligned} \ln\left|\frac{\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}}{x}\right| &= c_1 \\ \Rightarrow \ln\left|\frac{y + \sqrt{x^2 + y^2}}{x}\right| &= c_1 \\ \Rightarrow y + \sqrt{x^2 + y^2} &= c_2 x \end{aligned}$$

9.

$$y\frac{dx}{dy} + x(\ln x - \ln y - 1) = 0 \tag{5}$$

This equation is neither homogeneous nor is it a Bernoulli equation. However, since (5) can be written in the form

$$y\frac{dx}{dy} + \left(\ln\frac{x}{y} - 1\right) = 0 \tag{6}$$

it is natural to try the substitution

$$u = \frac{x}{y} \tag{7}$$

Then we have

$$\begin{array}{l} x = uy \\ \frac{dx}{dy} = u + y\frac{du}{dx} \end{array} . \tag{8}$$

Substitute (8) into (6):

$$\begin{aligned} y\left(u + y\frac{du}{dy}\right) + uy\left(\ln u - 1\right) &= 0 \\ \Rightarrow y\frac{du}{dy} + u\ln u &= 0 \end{aligned}$$

Separate variables and integrate:

$$\begin{aligned} \int \frac{\frac{1}{u}}{\ln u}du + \int \frac{dy}{y} &= c_1 \\ \Rightarrow \ln\left|\ln|u|\right| + \ln|y| &= c_1 \\ \Rightarrow \ln\left|y\ln|u|\right| &= c_1 \\ \Rightarrow y\ln|u| &= c \end{aligned}$$

Substitute (7):

$$y\ln\left|\frac{x}{y}\right| = c$$

Initial value: $x = 1\ ;\ y = e:$

$$c = e\ln\frac{1}{e} = -e$$

The solution is

$$y\ln\left|\frac{x}{y}\right| = -e$$

or

$$x = ye^{-\frac{e}{y}}.$$

10.

$$x + y + x\frac{dy}{dx} = 0 \tag{9}$$

Since

$$M(tx, ty) = tx + ty = t(x + y) = tM(x, y)$$

and

$$N(tx, ty) = tx = tN(x, y)$$

both functions are homogeneous functions of degree 1. We therefore let

$$y = ux \ ; \quad \frac{dy}{dx} = u + x\frac{du}{dx} \tag{10}$$

Substitute (10) in (9):

$$\begin{aligned} x + ux + x\left(u + x\frac{du}{dx}\right) &= 0 \\ \Rightarrow 1 + 2u + x\frac{du}{dx} &= 0 \end{aligned}$$

Separate variables and integrate:

$$\begin{aligned} \int \frac{dx}{x} + \int \frac{du}{1+2u} &= c_1 \\ \Rightarrow \ln|x| + \frac{1}{2}\ln|1+2u| &= c_1 \\ \Rightarrow 2\ln|x| + \ln|1+2u| &= c_2 \\ \Rightarrow \ln|x^2(1+2u)| &= c_2 \\ \Rightarrow x^2(1+2u) &= c \end{aligned}$$

Substitute $u = \frac{y}{x}$:

$$\begin{aligned} x^2\left(1 + 2\frac{y}{x}\right) &= c \\ \Rightarrow y(x) &= \frac{1}{2}\left(\frac{c}{x} - x\right). \end{aligned}$$

11.

$$\begin{aligned} 3\left(1+t^2\right)\frac{dy}{dt} &= 2ty(y^3 - 1) \\ \Rightarrow \frac{dy}{dt} + \frac{2t}{3(1+t^2)}y &= \frac{2t}{(1+t^2)}y^4 \end{aligned} \tag{11}$$

This is a Bernoulli equation with $n = 4$. We therefore let

$$u = u^{1-4} = \frac{1}{y^3} \tag{12}$$

Then follows that

$$\frac{du}{dt} = -\frac{3}{y^4}\frac{dy}{dt} = -\frac{3u}{y}\frac{dy}{dt}$$

and therefore

$$\frac{dy}{dt} = -\frac{y}{3u}\frac{du}{dt} \tag{13}$$

Substitute (12) and (13) in (11):

$$\begin{aligned} -\frac{y}{3u}\frac{du}{dt} + \frac{2t}{3(1+t^2)}y &= \frac{2t}{3(1+t^2)}\frac{y}{u} \\ \Rightarrow \frac{1}{u}\frac{du}{dt} + \frac{2t}{(1+t^2)}\left(\frac{1}{u} - 1\right) &= 0 \end{aligned}$$

Separate variables and integrate:

$$\begin{aligned} \int \frac{du}{1-u} + \int \frac{2tdt}{1+t^2} &= c_1 \\ \Rightarrow -\ln|1-u| + \ln(1+t^2) &= c_1 \\ \Rightarrow \ln\left|\frac{1+t^2}{1-u}\right| &= c_1 \\ \Rightarrow \frac{1+t^2}{1-u} &= c_2 \\ \Rightarrow 1-u &= c(1+t^2) \end{aligned}$$

Substitute $u = \frac{1}{y^3}$:

$$\begin{aligned} 1 - \frac{1}{y^3} &= c(1+t^2) \\ \Rightarrow y^3 &= \frac{1}{1-c(1+t^2)} \end{aligned}$$

12. We have the equation

$$(y^2 + xy)dx - x^2 dy = 0$$

where we identify $M(x,y) \equiv y^2 + xy$ and $N(x,y) \equiv -x^2$. Since

$$M(tx,ty) = t^2y^2 + txty = t^2(y^2 + xy) = t^2M(x,y)$$

is homogeneous of order 2, and

$$N(tx,ty) = -t^2x^2 = t^2(-x^2) = t^2N(x,y)$$

is homogeneous of order 2, we have a homogeneous differential equation. Thus we use the substitution

$$y(x) = u(x)x, \qquad \frac{dy}{dx} = u(x) + x\frac{du}{dx}, \qquad dy = u\,dx + x\,du$$

so that the equation

$$(y^2 + xy)dx - x^2dy = 0$$

becomes

$$(u^2x^2 + ux^2)dx - x^2(u\,dx + x\,du) = 0$$

and grouping differentials we find

$$u^2x^2dx = x^3du.$$

Integration yields

$$\int \frac{dx}{x} = \int \frac{du}{u^2} + c$$

$$\Rightarrow \ln|x| = -\frac{1}{u} + c = -\frac{x}{y} + c$$

since $u = y/x$. Thus the solution is given by

$$\frac{x}{y} + \ln|x| = c.$$

13. Rewriting the equation as

$$(y + \sqrt{x^2 - y^2})dx - xdy = 0$$

with $M(x,y) \equiv y + \sqrt{x^2 - y^2}$ and $N(x,y) \equiv -x$ we find that

$$M(tx,ty) = ty + \sqrt{t^2x^2 - t^2y^2} = t(y + \sqrt{x^2 - y^2}) = tM(x,y), \qquad t > 0$$

is of homogeneous of order 1 for $t \in [0, \infty)$, and

$$N(tx, ty) = -tx = tN(x, y)$$

is also homogeneous of order 1, so that the equation is a homogeneous differential equation. Using

$$y(x) = u(x)x, \qquad \frac{dy}{dx} = u(x) + x\frac{du}{dx}, \qquad dy = u\,dx + x\,du$$

we find

$$\left(ux + \sqrt{x^2 - u^2x^2}\right) dx - x(u\,dx + x\,du) = 0$$

$$\Rightarrow x\sqrt{1-u^2}dx = x^2du$$

since $x > 0$. Next we separate and integrate

$$\int \frac{du}{\sqrt{1-u^2}} = \int \frac{dx}{x} + c$$

$$\Rightarrow \arcsin(u) = \ln|x| + c$$

$$\Rightarrow \arcsin\left(\frac{y}{x}\right) = \ln|x| + c.$$

14. Using the substitution

$$u(x, y) \equiv x + y, \qquad \frac{du}{dx} = 1 + \frac{dy}{dx}$$

we obtain

$$\frac{du}{dx} - 1 = \frac{1-u}{u} = \frac{1}{u} - 1$$

which is trivially separated to obtain

$$\begin{aligned} \frac{1}{2}u^2 &= x + c \\ \Rightarrow (x+y)^2 &= 2x + 2c. \end{aligned}$$

15.

$$-ydx + (x + \sqrt{xy})\,dy = 0 \Rightarrow M(x, y) = -y \;\text{ and }\; N(x, y) = x + \sqrt{xy}$$

$$\begin{aligned}
M(tx,ty) &= -ty = t\,(-y) = tM(x,y) \\
N(tx,ty) &= tx + \sqrt{txty} = tx + \sqrt{t^2xy} = tx + t\sqrt{xy} = tN(x,y) \\
&\Rightarrow \text{ both } M \text{ and } N \text{ are homogeneous functions of degree 1.}
\end{aligned}$$

The substitution $y = ux$ gives

$$\begin{aligned}
-uxdx + \left(x + x\sqrt{u}\right)(udx + xdu) &= 0 \\
&\Rightarrow \left(x^2 + x^2\sqrt{u}\right) du + xu^{3/2}dx = 0 \\
&\Rightarrow \left(u^{-3/2} + \frac{1}{u}\right) du + \frac{dx}{x} = 0 \\
&\Rightarrow -2u^{-1/2} + \ln|u| + \ln|x| = c \\
&\Rightarrow \ln\left|\frac{y}{x}\right| + \ln|x| = 2\sqrt{\frac{x}{y}} + c \\
&\Rightarrow y\,(\ln|y| - c)^2 = 4x
\end{aligned}$$

16.

$$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{y^{-2}}{x} \Rightarrow \text{ Bernoulli DE with } n = -2$$

$$\text{So } \; u = y^{1-(-2)} = y^3 \; \left(\Rightarrow y = u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{u^{-2/3}}{3}\frac{du}{dx}\right)$$

$$\text{Hence, } \; \frac{u^{-2/3}}{3}\frac{du}{dx} + \frac{u^{1/3}}{x} = \frac{u^{-2/3}}{x} \Rightarrow \frac{du}{dx} + \left(\frac{3}{x}\right)u = \frac{3}{x}$$

which is a linear first-order DE with integrating factor $e^{\int \frac{3}{x}dx} = x^3$.

$$\begin{aligned}
&\Rightarrow x^3u = \int 3x^2dx = x^3 + c \\
&\Rightarrow x^3y^3 = x^3 + c \\
&\Rightarrow y^3 = 1 + cx^{-3}.
\end{aligned}$$

17.

$$x^2\frac{dy}{dx} - 2xy = 3y^4$$

$$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = \frac{3y^4}{x^2} \;\Rightarrow\; \text{Bernoulli DE with } n = 4$$

$$\text{So} \;\; u = y^{1-(4)} = y^{-3} \;\; \left(\Rightarrow y = u^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{u^{-4/3}}{3}\frac{du}{dx} \right)$$

$$\text{Hence,} \quad -\frac{u^{-4/3}}{3}\frac{du}{dx} - \frac{2u^{-1/3}}{x} = \frac{3u^{-4/3}}{x^2} \Rightarrow \frac{du}{dx} + \left(\frac{6}{x}\right) u = -\frac{9}{x^2}$$

which is a linear first-order DE with integrating factor $e^{\int \frac{6}{x} dx} = x^6$.

$$\Rightarrow \quad x^6 u = -9 \int x^4 dx = -\frac{9x^3}{5} + c$$

$$\Rightarrow \quad u = -\frac{9}{5x} + cx^{-6}$$

$$\Rightarrow \quad y^{-3} = -\frac{9}{5x} + cx^{-6}.$$

Initial value: $\quad y(1) = \frac{1}{2} \Rightarrow \left(\frac{1}{2}\right)^{-3} = -\frac{9}{5(1)} + c(1)$

$$\Rightarrow \quad 8 = -\frac{9}{5} + c$$

$$\Rightarrow \quad c = \frac{49}{5}$$

$$\Rightarrow \quad y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}.$$