TWK2A
Separable DEs
Solutions

1.

$$\begin{aligned} x\frac{dy}{dx} &= 4y \\ \int \frac{dy}{y} &= 4\int \frac{dx}{x}, \qquad y \neq 0, \quad x \neq 0 \\ \Rightarrow \ln|y| &= 4 \ \ln|x| + c \\ \Rightarrow y(x) &= e^c x^4 \end{aligned}$$

Of course, $y = 0$ is a singular solution. Moreover, since $x \neq 0$, the interval of definition for the solution must exclude 0.

2.

$$\begin{aligned} e^x y\frac{dy}{dx} &= e^{-y} + e^{-2x-y} = e^{-y}(1+e^{-2x}) \\ \Rightarrow \int ye^y dy &= \int (e^{-x} + e^{-3x})dx \\ \Rightarrow ye^y - e^y &= -e^{-x} - \frac{e^{-3x}}{3} + c \end{aligned}$$

It is acceptable to leave the solution in this implicit form, since further simplification is not possible.

3.

$$\begin{aligned} \frac{dP}{dt} &= P - P^2 \\ \Rightarrow \int \frac{dP}{P(1-P)} &= \int 1dt \\ \Rightarrow \int \left(\frac{1}{P} + \frac{1}{1-P}\right) dP &= t + c \\ \Rightarrow \ln|P| - \ln|1-P| &= t + c \\ \Rightarrow \frac{P}{1-P} &= e^c e^t \\ \Rightarrow P(t) &= \frac{e^c e^t}{1 + e^c e^t} \end{aligned}$$

4.

$$\begin{aligned}
\frac{d}{dx}(\ln(x^2+10)+\csc y) &= \frac{d}{dx}c \\
\Rightarrow \frac{2x}{x^2+10} - \csc y \cot y \frac{dy}{dx} &= 0 \\
\Rightarrow 2x\sin^2 y dx - (x^2+10)\cos y dy &= 0
\end{aligned}$$

where we multiplied by $\sin^2 y(x^2+10)$ to obtain the last equation. The implicit solution excludes $\sin y = 0$ (due to the $\csc y$ term) which suggests that

$$y = k\pi, \qquad k \in \mathbf{Z}.$$

are singular solutions to the equation.

5. Let the function be denoted by $y(x)$. The problem is described by the equation

$$\begin{aligned}
y^2 + (y')^2 &= 1 \\
\Rightarrow y' &= \pm\sqrt{1-y^2}, \qquad -1 \le y \le 1 \\
\Rightarrow \int \frac{dy}{\sqrt{1-y^2}} &= \pm\int dx, \qquad -1 < y < 1, \quad \sin u := y \\
\Rightarrow \int du &= \pm x + c \\
\Rightarrow u &= \pm x + c \\
\Rightarrow \sin u &= \sin(\pm x + c) \\
\Rightarrow y\,(x) &= \sin(\pm x + c)
\end{aligned}$$

Thus, for example, $y\,(x) = \sin x$ is a solution ($c = 0$) and $y\,(x) = \cos x$ is also a solution ($c = \frac{\pi}{2}$). We also find the constant solutions $y = 1$ and $y = -1$, which are, in fact, singular solutions.

6.

$$\begin{aligned}
\int \frac{dy}{(y-1)^2} - \int dx &= c \\
\Rightarrow -(y-1)^{-1} - x &= c \\
\Rightarrow \frac{1}{(y-1)} + x + c &= 0 \\
\Rightarrow y\,(x) &= 1 - \frac{1}{x+c}
\end{aligned}$$

7.

$$
\begin{aligned}
-\int \frac{dy}{(2y+3)^2} &= \int \frac{dx}{(4x+5)^2} + c \\
\Rightarrow -\frac{1}{2}(2y+3)^{-1} &= -\frac{1}{4}(4x+5)^{-1} + c
\end{aligned}
$$

Making y the subject of the equation, we find

$$y\,(x) = \frac{4x+5}{1-4c(4x+5)} - \frac{3}{2}$$

8.

$$
\begin{aligned}
\int \frac{dQ}{Q-70} &= \int k dt + c \\
\Rightarrow \ln|Q-70| &= kt + c \\
\Rightarrow Q = 70 + e^{kt+c} &= 70 + Ae^{kt} \quad \text{where } A := e^c
\end{aligned}
$$

9.

$$\frac{dy}{dt} = 1 - 2y$$

Separate variables and integrate:

$$\int \frac{dy}{1-2y} = \int dt + c \Rightarrow -\frac{1}{2}\ln|1-2y| = t + c$$

$$\Rightarrow 1 - 2y = e^{-2(t+c)} = Ae^{-2t}$$

where $A := e^{2c}$. Hence

$$y = \frac{1}{2}(1 - Ae^{-2t})$$

Substitute initial condition $t = 0 \ ; \ y = \frac{5}{2}$

$$\frac{5}{2} = \frac{1}{2}(1 - A) \ \Rightarrow A = -4$$

Solution:

$$y\,(x) = \frac{1}{2}(1 + 4e^{-2t})$$

10.

$$
\begin{aligned}
y\,dy &= x(1+x^2)^{-1/2}(1+y^2)^{1/2}dx \\
\Rightarrow \frac{y\,dy}{\sqrt{1+y^2}} &= \frac{x\,dx}{\sqrt{1+x^2}} \\
\Rightarrow \int \frac{y\,dy}{\sqrt{1+y^2}} &= \int \frac{x dx}{\sqrt{1+x^2}} + c \\
\Rightarrow \sqrt{1+y^2} &= \sqrt{1+x^2} + c
\end{aligned}
$$

11.

$$
\begin{aligned}
x^2 \frac{dy}{dx} &= y - xy \\
\Rightarrow x^2 \frac{dy}{dx} &= y(1-x) \\
\Rightarrow \frac{1}{y}\,dy &= \frac{1-x}{x^2}dx, \qquad y \neq 0, x \neq 0 \\
\Rightarrow \int \frac{1}{y}\,dy &= \int \frac{1-x}{x^2}dx + c \\
\Rightarrow \ln|y| &= -\frac{1}{x} - \ln|x| + c \\
\Rightarrow y\,(x) &= \frac{1}{x}e^{-\frac{1}{x}+c}
\end{aligned}
$$

We excluded $y = 0$ in this calculation. However, $y = 0$ does satisfy the differential equation but cannot satisfy the initial condition. Thus $y = 0$ is not a solution for the initial value problem. Substituting the initial value $x = -1$ and $y = -1$ allow us to determine c:

$$
-1 = -e^{1+c}, \qquad \Rightarrow c = -1.
$$

Thus the solution for the initial value problem is given by

$$
y\,(x) = \frac{1}{x}e^{-\frac{1}{x}-1}.
$$

12. Rearranging the equation

$$
\frac{dN}{dt} + N = Nte^{t+2}
$$

yields the separable form

$$\frac{dN}{dt} = N(te^{t+2} - 1).$$

Consequently we separate variables and integrate

$$\begin{aligned} \frac{1}{N}\frac{dN}{dt} &= te^{t+2} - 1 \\ \Rightarrow \int \frac{1}{N} dN &= \int (te^{t+2} - 1)dt + c \\ \Rightarrow \int \frac{1}{N} dN &= \int (e^2te^t - 1)dt + c \end{aligned}$$

To obtain the implicit solution

$$\ln|N| = e^2te^t - e^2e^t - t + c.$$

The solution is slightly easier to read in the form

$$\ln|N| = te^{t+2} - e^{t+2} - t + c.$$

13. Factorising yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy + 2y - x - 2}{xy - 3y + x - 3} = \frac{(x+2)y - (x+2)}{(x-3)y + (x-3)} \\ &= \frac{(x+2)(y-1)}{(x-3)(y+1)} = \frac{x+2}{x-3} \cdot \frac{y-1}{y+1} \end{aligned}$$

Thus we see $x \neq 3$ and $y \neq -1$ from the differential equation. Separating variables and integrating

$$\begin{aligned} \frac{dy}{dx} &= \frac{x+2}{x-3} \cdot \frac{y-1}{y+1} \\ \Rightarrow \frac{y+1}{y-1}\frac{dy}{dx} &= \frac{x+2}{x-3}, \qquad y \neq 1 \\ \Rightarrow \int \frac{y+1}{y-1} dy &= \int \frac{x+2}{x-3} dx + c \\ \Rightarrow \int \left(1 + \frac{2}{y-1}\right) dy &= \int \left(1 + \frac{5}{x-3}\right) dx + c \end{aligned}$$

yields the implicit solution

$$y + 2\ln|y-1| = x + 5\ln|x-3| + c.$$

Clearly $y(x) = 1$ is a (second) singular solution for the equation.

14.

$$\begin{aligned}
(e^x + e^{-x})\frac{dy}{dx} &= y^2 \\
\Rightarrow \frac{1}{y^2}\frac{dy}{dx} &= (e^x + e^{-x})^{-1}, \qquad y \neq 0 \\
\Rightarrow \int \frac{1}{y^2}dy &= \int (e^x + e^{-x})^{-1}dx + c \\
\Rightarrow -\frac{1}{y} &= \int \frac{e^x}{e^{2x}+1}dx + c, \qquad u(x) := e^x, \quad du = e^x dx \\
\Rightarrow -\frac{1}{y} &= \int \frac{du}{u^2+1} + c \\
\Rightarrow -\frac{1}{y} &= \arctan(u) + c = \arctan(e^x) + c \\
\Rightarrow y\,(x) &= \frac{-1}{\arctan(e^x) + c}, \qquad y \neq 0.
\end{aligned}$$

A singular solution is given by $y(x) = 0$.

15.

$$\begin{aligned}
x\frac{dy}{dx} &= y^2 - y \\
\Rightarrow \frac{1}{y(y-1)}\frac{dy}{dx} &= \frac{1}{x}, \qquad x \neq 0, y \neq 0, y \neq 1 \\
\Rightarrow \int \frac{1}{y(y-1)}dy &= \int \frac{1}{x}dx + c \\
\Rightarrow \int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy &= \ln|x| + c \\
\Rightarrow \ln|y-1| - \ln|y| &= \ln|x| + c \\
\Rightarrow \ln\left|\frac{y-1}{y}\right| &= \ln|x| + c
\end{aligned}$$

Obviously $y(x) = 0$ and $y(x) = 1$ are both singular solutions.

a) $x = 0, y = 1$: The solution is given by the singular solution $y(x) = 1$.
b) $x = 0, y = 0$: The solution is given by the singular solution $y(x) = 0$.
c) $x = \frac{1}{2}, y = \frac{1}{2}$:

$$-\ln\frac{1}{2} = c.$$

i.e.

$$\ln\left|\frac{y-1}{y}\right| = \ln|2x|.$$

d) $x = 2, y = \frac{1}{4}$:

$$\ln 3 = \ln 2 + c$$

i.e.

$$\ln\left|\frac{y-1}{y}\right| = \ln\left|\frac{3x}{2}\right|.$$