TWK2A
Linear DEs
Solutions

1.

$$
\begin{aligned}
3\frac{dy}{dx} + 12y &= 4 \\
\Rightarrow \quad \frac{dy}{dx} + 4y &= \frac{4}{3}
\end{aligned}
$$

Therefore $P(x) = 4$ and $f(x) = \frac{4}{3}$. We then have

$$\int P(x)dx = \int 4dx = 4x$$

and the solution is obtained from the equation

$$
\begin{aligned}
y\,(x) &= ce^{-\int Pdx} + e^{-\int Pdx} \int e^{\int Pdx} f(x)dx \\
\Rightarrow y\,(x) &= ce^{-4x} + e^{-4x} \int \frac{4}{3} e^{4x} dx \\
&= ce^{-4x} + \frac{1}{3}.
\end{aligned}
$$

Obviously the solution is defined for $-\infty < x < \infty$.

2.

$$
\begin{aligned}
x\frac{dy}{dx} &+ (1+x)y = e^{-x}\sin 2x \\
\Rightarrow \frac{dy}{dx} + (1+\frac{1}{x})y &= \frac{1}{x}e^{-x}\sin 2x
\end{aligned}
$$

Hence $P(x) = 1 + \frac{1}{x}$ and $f(x) = \frac{1}{x}e^{-x}\sin 2x$. Thus we have

$$\int P(x)dx = \int \left(1 + \frac{1}{x}\right) dx = x + \ln|x|$$

and so

$$e^{-\int Pdx} = e^{-x-\ln|x|} = \frac{e^{-x}}{x}.$$

The solution is obtained from the equation

$$\begin{aligned} y(x) &= ce^{-\int P dx} + e^{-\int P dx} \int e^{\int P dx} f(x) dx \\ &= \frac{ce^{-x}}{x} + \frac{e^{-x}}{x} \int xe^{x} \frac{1}{x} e^{-x} \sin 2x dx \\ &= \frac{ce^{-x}}{x} + \frac{e^{-x}}{x} \left(-\frac{1}{2} \cos 2x \right) \\ &= \frac{e^{-x}}{x} \left(c - \frac{1}{2} \cos 2x \right) \end{aligned}$$

The solution is defined on either $(-\infty, 0)$ or $(0, \infty)$. The solution is discontinuous at $x = 0$.

3.

$$\begin{aligned} (x+2)^2 \frac{dy}{dx} + (8+4x)y &= 5 \\ \Rightarrow \frac{dy}{dx} + \frac{4(2+x)}{(x+2)^2} y &= \frac{5}{(x+2)^2} \\ \Rightarrow \frac{dy}{dx} + \frac{4}{x+2} y &= \frac{5}{(x+2)^2} \end{aligned}$$

Hence

$$P(x) = \frac{4}{x+2} \quad ; \quad f(x) = \frac{5}{(x+2)^2}$$

and

$$\int P(x) dx = \int \frac{4}{x+2} dx = 4 \ln |x+2|$$

Hence,

$$e^{-\int P dx} = e^{-4 \ln |x+2|} = e^{\ln (x+2)^{-4}} = (x+2)^{-4}$$

and the solution follows from the equation

$$\begin{aligned} y(x) &= ce^{-\int P dx} + e^{-\int P dx} \int e^{\int P dx} f(x) dx \\ &= c(x+2)^{-4} + (x+2)^{-4} \int (x+2)^4 \frac{5}{(x+2)^2} dx \\ &= c(x+2)^{-4} + (x+2)^{-4} \int 5(x+2)^2 dx \\ &= c(x+2)^{-4} + (x+2)^{-4} \frac{5}{3}(x+2)^3 \\ &= \frac{c}{(x+2)^4} + \frac{5}{3(x+2)} \end{aligned}$$

The solution exists for $-2 < x < \infty$.

4. NB: In this DE x is the dependent variable and y is the independent variable.

$$\frac{dx}{dy} - \frac{1}{y}x = 2y$$

Thus

$$P(y) = -\frac{1}{y}; \quad f(y) = 2y$$

and so

$$\int P(y) dy = -\int \frac{1}{y} dy = -\ln|y|.$$

Clearly, $y \neq 0$. The integrating factor is

$$e^{-\ln|y|} = \frac{1}{y}$$

and the solution follows from the equation

$$\frac{d}{dy}\left[x\frac{1}{y}\right] = \frac{1}{y}(2y) = 2.$$

Integrate wrt y :

$$\frac{x}{y} = c + 2y \Rightarrow x(y) = cy + 2y^2$$

The solution exists for $0 < y < \infty$. Initial values:

$$x = 1; \quad y = 5.$$

Thus

$$1 = 5c + 50 \quad ; \quad c = \frac{49}{5}$$

Solution:

$$x\left(y\right) = 2y^2 - \frac{49}{5}y.$$

5.

$$\begin{aligned} x^2y' + xy &= 1 \\ \Rightarrow y' + \frac{1}{x}y &= \frac{1}{x^2} \end{aligned}$$

Therefore

$$P(x) = \frac{1}{x} \Rightarrow \int P(x)dx = \int \frac{1}{x}dx = \ln|x|$$

and

$$e^{-\int P(xdx)} = e^{-\ln|x|} = \frac{1}{x}.$$

The solution then follows from the equation

$$\begin{aligned} y\left(x\right) &= ce^{-\int Pdx} + e^{-\int Pdx}\int e^{\int Pdx}f(x)dx \\ &= \frac{c}{x} + \frac{1}{x}\int x\frac{1}{x^2}dx \\ &= \frac{c}{x} + \frac{1}{x}\ln|x| \end{aligned}$$

The solution is defined on either $(-\infty, 0)$ or $(0, \infty)$.

6.

$$\begin{aligned} x^2y' + x(x+2)y &= e^x \\ \Rightarrow y' + \left(1 + \frac{2}{x}\right)y &= \frac{e^x}{x^2} \end{aligned}$$

Therefore

$$\begin{aligned} P\quad(x) =& \ 1 + \frac{2}{x} \\ \Rightarrow \int P(x)dx \quad =& \quad \int \left(1 + \frac{2}{x}\right)dx = x + 2\ln|x| = x + \ln x^2 \end{aligned}$$

and so

$$e^{-\int P(x)dx} = e^{-x-\ln x^2} = x^{-2}e^{-x}.$$

The solution then follows from the equation

$$\begin{aligned}
y(x) &= ce^{-\int Pdx} + e^{-\int Pdx}\int e^{\int Pdx} f(x)dx \\
&= cx^{-2}e^{-x} + x^{-2}e^{-x}\int x^2e^x\frac{e^x}{x^2}dx \\
&= cx^{-2}e^{-x} + x^{-2}e^{-x}\frac{1}{2}e^{2x} \\
&= cx^{-2}e^{-x} + \frac{1}{2}e^x x^{-2}
\end{aligned}$$

The solution is defined on either $(-\infty, 0)$ or $(0, \infty)$.

7.

$$\begin{aligned}
x\frac{dy}{dx} + (3x+1)y &= e^{-3x} \\
\Rightarrow \frac{dy}{dx} + \left(3 + \frac{1}{x}\right)y &= \frac{e^{-3x}}{x}
\end{aligned}$$

Therefore

$$P(x) = 3 + \frac{1}{x} \Rightarrow \int P(x)dx = 3x + \ln|x|$$

and

$$e^{-\int P(x)dx} = e^{-3x-\ln|x|} = x^{-1}e^{-3x}.$$

The solution then follows from the equation

$$\begin{aligned}
y(x) &= ce^{-\int Pdx} + e^{-\int Pdx}\int e^{\int Pdx} f(x)dx \\
&= cx^{-1}e^{-3x} + x^{-1}e^{-3x}\int xe^{3x}\frac{e^{-3x}}{x}dx \\
&= cx^{-1}e^{-3x} + x^{-1}e^{-3x}\int dx \\
&= cx^{-1}e^{-3x} + x^{-1}e^{-3x}x \\
&= cx^{-1}e^{-3x} + e^{-3x}
\end{aligned}$$

The solution is defined on either $(-\infty, 0)$ or $(0, \infty)$.

8.

$$\begin{aligned}(x+1)\frac{dy}{dx}+y &= \ln x\\ \Rightarrow \frac{dy}{dx}+\frac{1}{x+1}y &= \frac{\ln x}{1+x}\end{aligned}$$

Therefore

$$P(x)=\frac{1}{x+1}\Rightarrow \int P(x)dx=\ln|x+1|$$

and the integrating factor is

$$e^{\int P(x)\,dx}=e^{\ln|x+1|}=x+1.$$

The solution follows from the equation

$$\frac{d}{dx}\left[(x+1)y\right]=\frac{\ln x}{1+x}(1+x)=\ln x$$

Integrate wrt x:

$$(x+1)y=x\ln x-x+c$$

Substitute the initial value $y(1)=10$:

$$2.10=0-1+c\ \Rightarrow c=21$$

The solution to the initial-value problem is

$$\begin{aligned}(x+1)y &= x\ln x-x+21\\ \Rightarrow y\left(x\right) &= \frac{x\ln x-x+21}{x+1}\end{aligned}$$

Since $\ln x$ is only defined for positive x and $\frac{1}{x+1}$ exists for all x except $x=-1$, the solution is defined on $(0,\infty)$.

9. The equation is already in standard form with $P(x)=2x$. Thus we use

$e^{-\int 2xdx} = e^{-x^2}$ in the equation

$$\begin{aligned} y(x) &= ce^{-\int Pdx} + e^{-\int Pdx}\int e^{\int Pdx} f(x)dx \\ &= ce^{-x^2} + e^{-x^2}\int e^{x^2}x^3dx \\ &= ce^{-x^2} + e^{-x^2}\left[\frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx\right] \\ &= ce^{-x^2} + e^{-x^2}\left[\frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}\right] \\ &= ce^{-x^2} + \frac{1}{2}x^2 - \frac{1}{2}. \end{aligned}$$

Solution is defined on $(-\infty, \infty)$. In evaluating the integral we have used the substitution $u = x^2$, which implies $du = 2xdx$. Hence, $\int e^{x^2}x^3dx = \frac{1}{2}\int ue^u du = \frac{1}{2}ue^u - \frac{1}{2}\int e^u du = \frac{1}{2}ue^u - \frac{1}{2}e^u = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2}$. Note that in the third line above, the integral $\int xe^{x^2}dx$ arises from $\frac{1}{2}\int e^u du$, since $du = 2xdx \Rightarrow \frac{du}{2} = xdx$

10. This equation is non-linear in y, but linear in x. In standard form

$$\frac{dx}{dy} + \frac{2}{y}x = e^y, \qquad y \neq 0$$

we find (with $P(y) = \frac{2}{y}$)

$$\begin{aligned} x(y) &= ce^{-\int Pdy} + e^{-\int Pdy}\int e^{\int Pdy} f(y)dy \\ &= cy^{-2} + y^{-2}\int y^2e^y dy \end{aligned}$$

Using integration by parts we find

$$\int y^2e^ydy + c = y^2e^y - \int 2ye^ydy + c = y^2e^y - 2ye^y + \int 2e^ydy + c = y^2e^y - 2ye^y + 2e^y + c$$

so that (on $(0, \infty)$ or on $(-\infty, 0)$)

$$x(y) = e^y - \frac{2}{y}e^y + \frac{2}{y^2}e^y + \frac{c}{y^2}.$$

From the DE

$$y = (ye^y - 2x)\frac{dy}{dx}$$

we see that $y(x) = 0$ is a singular solution.

11. In standard form

$$\frac{dy}{dx} + \frac{x+2}{x+1}y = \frac{2x}{x+1}e^{-x}, \qquad x \neq -1$$

so that

$$e^{-\int \frac{x+2}{x+1}dx} = e^{-1}e^{-x-\ln|x+1|} = e^{-1}|x+1|^{-1}e^{-x}.$$

Since $x \neq -1$ we need to decide on our domain for the rest of the derivation, i.e. $(-\infty, -1)$ or $(-1, \infty)$. We choose the latter (normally the choice depends on the initial value problem) to obtain

$$e^{-\int \frac{x+2}{x+1}dx} = e^{-1}(x+1)^{-1}e^{-x}.$$

Hence,

$$\begin{aligned} y(x) &= ce^{-\int Pdx} + e^{-\int Pdx}\int e^{\int Pdx} f(x)dx \\ &= c(x+1)^{-1}e^{-x} + e^{-1}(x+1)^{-1}e^{-x}\int e(x+1)e^{x}\frac{2x}{x+1}e^{-x}dx \\ &= c(x+1)^{-1}e^{-x} + (x+1)^{-1}e^{-x}\int 2xdx \\ &= c(x+1)^{-1}e^{-x} + (x+1)^{-1}e^{-x}x^2 \\ &= \frac{e^{-x}}{x+1}\left(c + x^2\right) \end{aligned}$$

defined on $(-1, \infty)$.

12. The equation is already in standard form with $P(x) = \tan x$ so that

$$u(x) = e^{\int \tan x dx} = e^{-\ln|\cos x|} = \frac{1}{|\cos x|}.$$

Note that the differential equation already implies $\cos(x) \neq 0$ (due to the $\tan x$ term), so the interval of definition excludes these values of x. Furthermore, $\cos(x)$ can obviously not change sign on the interval of definition, since that would imply that it passes through zero. Thus we may choose, for example $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. This interval includes the intial value $x = 0$. On this interval $u(x) = \frac{1}{\cos x}$. Multiplying the differential equation by $u(x)$ yields

$$\frac{1}{\cos x}\frac{dy}{dx} + \frac{\sin x}{\cos^2 x}y = \cos x \Rightarrow \frac{d}{dx}\left[\frac{1}{\cos x}y\right] = \cos x$$

which can be integrated to find

$$\frac{y}{\cos x} = \sin x + c.$$

Inserting the initial value $x = 0$, $y = -1$ we find $c = -1$. Thus the solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ is

$$y(x) = \cos x(\sin x - 1).$$

13. The form of the solution to the differential equation

$$\frac{dy}{dx} + P(x)y = f(x)$$

is

$$y(x) = \frac{1}{u(x)}\left(\int u(x)f(x)dx + c\right)$$

where $u(x) \neq 0$ and

$$\frac{du}{dx} = u(x)P(x).$$

Thus we find

$$u(x) = e^{\int P(x)dx+k} = e^k e^{\int P(x)dx}$$

where k is the constant of integration. Inserting into the solution yields

$$y(x) = e^{-k}e^{-\int P(x)dx}\left(\int e^k e^{\int P(x)dx} f(x)dx + c\right)$$

and applying the distributive rule

$$\begin{aligned} y(x) &= e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)dx + ce^{-k}e^{-\int P(x)dx} \\ &= e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)dx + c_1 e^{-\int P(x)dx} \\ &= e^{-\int P(x)dx}\left(\int e^{\int P(x)dx} f(x)dx + c_1\right) \end{aligned}$$

where $c_1 := ce^{-k}$ is an arbitrary constant.

14. Consider the first equation and initial condition

$$\frac{dx}{dt} = -\lambda_1 x, \qquad x(0) = x_0.$$

This is an initial value problem where the equation is separable

$$\int \frac{dx}{x} = -\int \lambda_1 dt + c$$

$$\ln|x| = -\lambda_1 t + c.$$

Assuming $x > 0$ and $x_0 > 0$ we obtain $x = e^c e^{-\lambda_1 t}$. The initial condition $t = 0, x = x_0$ yields $e^c = x_0$ so that the solution is given by $x(t) = x_0 e^{-\lambda_1 t}$. Inserting into the second equation we have the initial value problem

$$\begin{aligned} \frac{dy}{dt} &= \lambda_1 x_0 e^{-\lambda_1 t} - \lambda_2 y, \qquad y(0) = y_0 \\ \Rightarrow \frac{dy}{dt} + \lambda_2 y &= \lambda_1 x_0 e^{-\lambda_1 t} \end{aligned}$$

which is a linear equation. The integrating factor is $u(t) = e^{\lambda_2 t}$. Thus we have

$$\frac{d}{dt}\left[e^{\lambda_2 t} y\right] = \lambda_1 x_0 e^{(\lambda_2 - \lambda_1)t}.$$

- in other words, for $\lambda_2 \neq \lambda_1$,

  $$e^{\lambda_2 t} y = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + k_1.$$

  Inserting the intial condition $t = 0, y = y_0$ gives

  $$y_0 = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} + k_1$$

  so that the final solution for $y(t)$ is

  $$y = \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \left(y_0 - \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} \right) e^{-\lambda_2 t}.$$

- When $\lambda_2 = \lambda_1$ we have the equation
$$\frac{d}{dt}\left[e^{\lambda_2 t} y\right] = \lambda_1 x_0,$$
with the solution
$$y = \lambda_1 x_0 t e^{-\lambda_2 t} + k_2 e^{-\lambda_2 t}.$$
Inserting the initial condition $t = 0, y = y_0$ gives the solution $k_2 = y_0$,
$$y = \lambda_1 x_0 t e^{-\lambda_2 t} + y_0 e^{-\lambda_2 t}.$$

Thus we find the solution

$$\begin{aligned} x(t) &= x_0 e^{-\lambda_1 t}, \\ y(t) &= \begin{cases} \lambda_1 x_0 t e^{-\lambda_2 t} + y_0 e^{-\lambda_2 t} & \lambda_2 = \lambda_1 \\ \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \left(y_0 - \frac{\lambda_1 x_0}{\lambda_2 - \lambda_1}\right) e^{-\lambda_2 t} & \lambda_2 \neq \lambda_1 \end{cases} \end{aligned}$$

15.

$$\begin{aligned} \frac{dy}{dx} + y &= e^{-3x} \Rightarrow P(x) = 1 \Rightarrow e^{-\int P dx} = e^{-x} \\ \Rightarrow y\left(x\right) &= c e^{-\int P dx} + e^{-\int P dx} \int e^{\int P dx} f(x) dx \\ &= c e^{-x} + e^{-x} \int e^{x} e^{-3x} dx \\ &= c e^{-x} + e^{-x} \left(-\frac{e^{-2x}}{2}\right) \\ &= c e^{-x} - \frac{e^{-3x}}{2} \qquad I = (-\infty, \infty) \end{aligned}$$

16.

$$\begin{aligned} y' + 3x^2 y &= 10x^2 \Rightarrow P(x) = 3x^2 \Rightarrow e^{-\int P dx} = e^{-x^3} \\ \Rightarrow y\left(x\right) &= c e^{-\int P dx} + e^{-\int P dx} \int e^{\int P dx} f(x) dx \\ &= c e^{-x^3} + e^{-x^3} \int e^{x^3} 10x^2 dx \\ &= c e^{-x^3} + e^{-x^3} \frac{10}{3} e^{x^3} \\ &= c e^{-x^3} + \frac{10}{3} \qquad I = (-\infty, \infty) \end{aligned}$$

17.

$$\begin{aligned}
\cos x \frac{dy}{dx} + y\sin x &= 1 \Rightarrow \frac{dy}{dx} + y\tan x = \sec x \\
\Rightarrow P(x) &= \tan x \Rightarrow \int P dx = -\ln|\cos x| \\
\Rightarrow e^{-\int P dx} &= |\cos x|
\end{aligned}$$

$$\begin{aligned}
\Rightarrow y(x) &= ce^{-\int P dx} + e^{-\int P dx}\int e^{\int P dx} f(x)dx \\
&= c|\cos x| + |\cos x|\int \frac{\sec x}{|\cos x|}dx \\
&= c|\cos x| + |\cos x|\int \sec^2 x dx \\
&= c|\cos x| + |\cos x|\tan x
\end{aligned}$$

Defined on any interval between points where $\tan x$ is undefined $\left(x \neq \frac{\pi}{2} + n\pi \ ; \ n \in \mathbb{Z}\right)$ e.g. $I = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

18.

$$\left(x^2 - 1\right)\frac{dy}{dx} + 2y = (x+1)^2 \Rightarrow \frac{dy}{dx} + \frac{2y}{(x^2-1)} = \frac{(x+1)^2}{(x-1)(x+1)} = \left(\frac{x+1}{x-1}\right)$$

$$\Rightarrow \quad P(x) = \frac{2}{(x^2-1)}$$

$$\begin{aligned}
\Rightarrow \int P dx &= \ln\left|\frac{x-1}{x+1}\right| \qquad \text{(see \#44 in table of integrals in Z\&C 6th ed.)} \\
\Rightarrow e^{-\int P dx} &= \left|\frac{x+1}{x-1}\right| \\
\Rightarrow y(x) &= ce^{-\int P dx} + e^{-\int P dx}\int e^{\int P dx} f(x)dx \\
&= c\left|\frac{x+1}{x-1}\right| + \left|\frac{x+1}{x-1}\right|\int\left|\frac{x-1}{x+1}\right|\left(\frac{x+1}{x-1}\right)dx \\
&= c\left|\frac{x+1}{x-1}\right| + \left|\frac{x+1}{x-1}\right|\left(\pm\int dx\right) \\
&= c\left|\frac{x+1}{x-1}\right| \pm x\left|\frac{x+1}{x-1}\right|.
\end{aligned}$$

Note that $y(x)$ is not defined when $x = 1$. So $I = (-\infty, 1)$ or $(1, \infty)$.