TWK2A
Exact DEs
Solutions

1.

$$\begin{aligned}\frac{\partial}{\partial y}(\sin y - y\sin x) &= \cos y - \sin x \\ \frac{\partial}{\partial x}(\cos x + x\cos y - y) &= -\sin x + \cos y\end{aligned}$$

Thus the equation is exact.

$$\begin{aligned}f(x,y) &\equiv \int (\sin y - y\sin x)dx + g(y) \\ &= x\sin y + y\cos x + g(y) \\ \frac{\partial f}{\partial y} &= x\cos y + \cos x + \frac{dg}{dy} \\ &= \cos x + x\cos y - y \\ \frac{dg}{dy} &= -y \Rightarrow g(y) = -\frac{1}{2}y^2 + c \\ f(x,y) &= x\sin y + y\cos x - \frac{1}{2}y^2 + c = 0\end{aligned}$$

2.

$$\begin{aligned}\frac{\partial}{\partial y}(1 + \ln x + \frac{y}{x}) &= \frac{1}{x} \\ \frac{\partial}{\partial x}(\ln x - 1) &= \frac{1}{x}\end{aligned}$$

Thus the equation is exact.

$$\begin{aligned}f(x,y) &\equiv \int (\ln x - 1)dy + g(x) \\ &= y\ln x - y + g(x) \\ \frac{\partial f}{\partial x} &= \frac{y}{x} + \frac{dg}{dx} \\ &= 1 + \ln x + \frac{y}{x} \\ \frac{dg}{dx} &= 1 + \ln x \Rightarrow g(x) = x\ln x + c \\ f(x,y) &= y\ln x - y + x\ln x + c = 0\end{aligned}$$

3.

$$\begin{aligned}
\frac{\partial}{\partial y}(4t^3y - 15t^2 - y) &= 4t^3 - 1 \\
\frac{\partial}{\partial t}(t^4 + 3y^2 - t) &= 4t^3 - 1
\end{aligned}$$

Thus the equation is exact.

$$\begin{aligned}
f(t,y) &\equiv \int (t^4 + 3y^2 - t)dy + g(t) \\
&= yt^4 + y^3 - ty + g(t) \\
\frac{\partial f}{\partial t} &= 4yt^3 - y + \frac{dg}{dt} \\
&= 4t^3y - 15t^2 - y \\
\frac{dg}{dt} &= -15t^2 \Rightarrow g(t) = -\frac{15}{3}t^3 + c \\
f(t,y) &= yt^4 + y^3 - ty - \frac{15}{3}t^3 + c = 0
\end{aligned}$$

4.

$$\begin{aligned}
\frac{\partial}{\partial y}(x+y)^2 &= 2(x+y) \\
\frac{\partial}{\partial x}(2xy + x^2 - 1) &= 2y + 2x
\end{aligned}$$

Thus the equation is exact.

$$\begin{aligned}
f(x,y) &\equiv \int (2xy + x^2 - 1)dy + g(x) \\
&= xy^2 + x^2y - y + g(x) \\
\frac{\partial f}{\partial x} &= y^2 + 2xy + \frac{dg}{dx} \\
&= x^2 + 2xy + y^2 \\
\frac{dg}{dx} &= x^2 \Rightarrow g(x) = \frac{x^3}{3} + c
\end{aligned}$$

$$
\begin{aligned}
f(x,y) &= xy^2 + x^2y - y + \frac{x^3}{3} + c = 0 \\
y(1) &= 1 \Rightarrow 1 + 1 - 1 + \frac{1}{3} + c = 0 \Rightarrow c = -\frac{4}{3} \\
f(x,y) &= xy^2 + x^2y - y + \frac{x^3}{3} - \frac{4}{3} = 0
\end{aligned}
$$

5. From the hint and using the substitution $w(x,y) = x^2 + y^2$ and $dw = 2(xdx + ydy)$

$$
\begin{aligned}
(x - \sqrt{x^2+y^2})dx + ydy &= 0 \\
\Rightarrow \frac{xdx + ydy}{\sqrt{x^2+y^2}} &= dx \\
\Rightarrow \int \frac{dw}{2\sqrt{w}} &= \int dx \qquad (w \equiv x^2 + y^2) \\
\Rightarrow \sqrt{w} &= x + c \\
\Rightarrow x^2 + y^2 &= (x+c)^2
\end{aligned}
$$

6.

$$
\begin{aligned}
\frac{dy}{dx} &= g(x)h(y) \\
&\Rightarrow \left(\frac{1}{h(y)}\right) dy + (-g(x))\, dx = 0 \\
&\Rightarrow M(x,y) = -g(x) \quad \text{and} \quad N(x,y) = \frac{1}{h(y)}
\end{aligned}
$$

Clearly

$$
\frac{\partial M}{\partial y} = 0 \quad \text{and} \quad \frac{\partial N}{\partial x} = 0
$$

since M is a function of y only, and N is a function of x only. Hence, the DE is exact.

7.

$$
\left(1 - \frac{3}{y} + x\right) \frac{dy}{dx} + y = \frac{3}{x} - 1
$$

We rearrange the ODE:

$$\begin{aligned} \left(y+1-\frac{3}{x}\right)+\left(1-\frac{3}{y}+x\right)\frac{dy}{dx} &= 0 \\ \Rightarrow \left(y+1-\frac{3}{x}\right)dx+\left(1-\frac{3}{y}+x\right)dy &= 0 \end{aligned}$$

Hence,

$$M = y+1-\frac{3}{x} \quad ; \quad N = 1-\frac{3}{y}+x$$

Since

$$\frac{\partial M}{\partial y} = 1 \quad ; \quad \frac{\partial N}{\partial x} = 1$$

the ODE is exact. Therefore $f(x,y)$ exists such that

$$\begin{aligned} \frac{\partial f}{\partial x} &= M = y+1-\frac{3}{x} \\ f &= \int \left(y+1-\frac{3}{x}\right)dx + g(y) = xy + x - 3\ln|x| + g(y) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= N = 1-\frac{3}{y}+x \\ f &= \int \left(1-\frac{3}{y}+x\right)dy + h(x) = y - 3\ln|y| + xy + h(x) \end{aligned}$$

By inspection we establish that

$$g = y - 3\ln|y| \quad ; \quad h = x.$$

Hence

$$\begin{aligned} f &= x+y+xy-3\ln|x|-3\ln|y| \\ &= x+y+xy-3\ln|xy| \\ &= x+y+xy-\ln|xy|^3 \end{aligned}$$

and the solution is given by

$$x+y+xy-\ln|xy|^3 = c$$

8. The DE has the form

$$M(x,y)dx + N(x,y)dy = 0$$

with

$$M = e^x + y \quad ; \quad N = 2 + x + ye^y$$

Hence,

$$\frac{\partial M}{\partial y} = 1 \quad ; \quad \frac{\partial N}{\partial x} = 1$$

and so the DE is exact. Thus there exists $f(x,y)$ such that

$$\begin{aligned} \frac{\partial f}{\partial x} &= M = e^x + y \\ f &= \int (e^x + y)\, dx + g(y) = e^x + xy + g(y) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= N = 2 + x + ye^y \\ f &= \int (2 + x + ye^y) dy + h(x) \\ &= 2y + xy + ye^y - \int e^y\, dy + h(x) \\ &= 2y + xy + ye^y - e^y + h(x) \end{aligned}$$

By inspection we find

$$g = 2y + (y-1)e^y \qquad\qquad h = e^x$$

and so

$$f = e^x + 2y + xy + (y-1)e^y.$$

The solution is thus given by

$$e^x + 2y + xy + (y-1)e^y = c. \tag{1}$$

Initial-value: $x = 0 \;\; ; \;\; y = 1$

$$1 + 2 + 0 + 0 = c \quad ; \quad c = 3$$

Equation (1) becomes

$$e^x + 2y + xy + (y-1)e^y = 3.$$

9.

$$y(x+y+1)dx + (x+2y)dy = 0 \tag{2}$$

With

$$M \equiv y(x+y+1) \quad ; \quad N \equiv x+2y$$

we have

$$\frac{\partial M}{\partial y} = x+2y+1 \quad ; \quad \frac{\partial N}{\partial x} = 1.$$

Therefore, the ODE is not exact. However, we also have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x+2y+1-1}{x+2y} = 1.$$

Therefore, the ODE can be made exact with the help of the integrating factor

$$\mu(x) = e^{\int 1 dx} = e^x.$$

Multiply (2) by $\mu(x)$:

$$ye^x(x+y+1)dx + e^x(x+2y)dy = 0 \tag{3}$$

The DE (3) now is exact.

[**CHECK** :

With $M = ye^x(x+y+1) \quad ; \quad N = e^x(x+2y)$

we have $\dfrac{\partial M}{\partial y} = e^x(x+2y+1) \quad ; \quad \dfrac{\partial N}{\partial x} = e^x(x+2y+1)$]

Therefore, $f(x,y)$ exists such that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= ye^x(x+y+1) \\
f &= \int ye^x(x+y+1)dx + g(y) \\
&= y\int xe^x dx + y(y+1)\int e^x dx + g(y) \\
&= y(xe^x - \int e^x dx) + y(y+1)e^x + g(y) \\
&= yxe^x - ye^x + y(y+1)e^x + g(y) \\
&= ye^x(x+y) + g(y)
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial y} &= e^x(x+2y) \\ f &= \int e^x(x+2y)dy + h(x) \\ &= e^x(xy+y^2)+h(x)\end{aligned}$$

From inspection, $g = h = 0$. Hence

$$f = ye^x(x+y)$$

and the solution to (2) is given by

$$ye^x(x+y) = c.$$

10.

$$\cos x\, dx + \left(1+\frac{2}{y}\right)\sin x\, dy = 0 \tag{4}$$

With

$$M \equiv \cos x \quad ; \quad N \equiv \left(1+\frac{2}{y}\right)\sin x$$

we have

$$\frac{\partial M}{\partial y} = 0 \quad ; \quad \frac{\partial N}{\partial x} = \left(1+\frac{2}{y}\right)\cos x$$

So the DE is not exact. Since

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{\left(1+\frac{2}{y}\right)\cos x}{\cos x} = 1+\frac{2}{y}$$

we find an integrating factor:

$$\mu(y) = e^{\int\left(1+\frac{2}{y}\right)dy} = e^{y+2\ln|y|} = y^2e^y.$$

Multiplying (4) by $\mu(y)$:

$$y^2e^y\cos\, dx + y^2e^y\left(1+\frac{2}{y}\right)\sin x\, dy = 0 \tag{5}$$

Equation (2) is now exact.

[**CHECK** :

$$\text{With } M = y^2 e^y \cos x \ ; \quad N = e^y(y^2+2y)\sin x$$

$$\text{we have} \quad \frac{\partial M}{\partial y} = (2y+y^2)e^y \cos x \ ; \quad \frac{\partial N}{\partial x} = e^y(y^2+2y)\cos x \Big]$$

Thus there exists $f(x,y)$ such that

$$\begin{aligned} \frac{\partial f}{\partial x} &= y^2 e^y \cos x \\ f &= y^2 e^y \int \cos x dx + g(y) = y^2 e^y \sin x + g(h) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= e^y(y^2+2y)\sin x \\ f &= \sin x \int (y^2 e^y + 2ye^y) dy + h(x) \\ &= \sin x \left(y^2 e^y - 2\int ye^y dy + 2\int ye^y dy \right) + h(x) \\ &= y^2 e^y \sin x + h(x) \end{aligned}$$

By inspection $g = h = 0$, and so

$$f = y^2 e^y \sin x$$

The solution of (4) is

$$y^2 e^y \sin x = c$$

11.

$$(x^2+y^2-5)\,dx - (y+xy)\,dy = 0 \qquad (6)$$

With

$$M \equiv x^2y^2 - 5 \ ; \quad N \equiv -(y+xy)$$

we have

$$\frac{\partial M}{\partial y} = 2y \ ; \quad \frac{\partial N}{\partial x} = -y$$

Therefore the DE (6) is not exact. However, since

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-(-y)}{-(y+xy)} = \frac{3y}{-y(1+x)} = -\frac{3}{1+x},$$

an integrating factor is given by

$$\mu(x) = e^{\int\left(-\frac{3}{1+x}\right)dx} = e^{-3\ln|1+x|} = \frac{1}{(1+x)^3}$$

Multiply (6) by μ:

$$\frac{x^2+y^2-5}{(1+x)^3}dx - \frac{y}{(1+x)^2}dy = 0 \tag{7}$$

It is left to the student to verify that (7) indeed is exact. There now exists $f(x,y)$ such that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{x^2+y^2-5}{(1+x)^3} \\
f &= \int \left[x^2(1+x)^{-3} + (y^2-5)(1+x)^{-3}\right] dx + g(y) \\
&= -\frac{1}{2}x^2(1+x)^{-2} - \int 2x\left(-\frac{1}{2}\right)(1+x)^{-2}dx - \frac{1}{2}(y^2-5)(1+x)^{-2} + g(y) \\
&= -\frac{1}{2}x^2(1+x)^{-2} + (-1)x(1+x)^{-1} - \int(-1)(2+x)^{-1}dx - \frac{1}{2}(y^2-5)(1+x)^{-2} \\
&= -\frac{x^2}{2(1+x)^2} - \frac{x}{1+x} + \ln|1+x| - \frac{y^2-5}{2(1+x)^2} + g(y)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial y} &= -\frac{y}{(1+x)^2} \\
f &= -\frac{1}{(1+x)^2}\int y\,dy + h(x) = -\frac{y^2}{2(1+x)^2} + h(x)
\end{aligned}$$

From inspection we have

$$\begin{aligned}
g(y) &= 0 \\
h(x) &= -\frac{x^2}{2(1+x)^2} - \frac{x}{1+x} + \ln|1+x| + \frac{5}{2(1+x)^2}
\end{aligned}$$

Therefore, the solution is given by

$$\frac{x^2}{2(1+x)^2}+\frac{x}{1+x}-\ln|1+x|+\frac{y^2-5}{2(1+x)^2}=c. \tag{8}$$

We substitute the initial condition: $x=0 \;\; ; \;\; y=1$

$$0+0-0+\frac{1-5}{2}=c \;\; ; \;\; c=-2$$

Equation (8) then becomes

$$\frac{x^2+y^2-5}{2(1+x)^2}+\frac{x}{1+x}-\ln|1+x|=-2$$

12.

$$(\tan x-\sin x\sin y)dx+\cos x\cos y dy=0$$

Criteria for exactness:

$$\begin{aligned} M &= \tan x-\sin x\sin y \;\; ; & \tfrac{\partial M}{\partial y}&=-\sin x\cos y \\ N &= \cos x\cos y \;\; ; & \tfrac{\partial N}{\partial x}&=-\sin x\cos y \end{aligned}$$

Thus $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ and the DE is exact.

$$\begin{aligned} \tfrac{\partial f}{\partial x} &= \tan x-\sin x\sin y & \Rightarrow \quad & f=-\ln|\cos x|+\cos x\sin y+g(y) \\ \tfrac{\partial f}{\partial y} &= \cos x\cos y & \Rightarrow \quad & f=\cos x\sin y+h(x) \end{aligned}$$

Thus $g(y)=0$; $h(x)=-\ln|\cos x|$ and

$$f=\cos x\sin y-\ln|\cos x|$$

The solution is given by

$$\cos x\sin y-\ln|\cos x|=c$$

13.

$$\left(y^2\cos x-3x^2y-2x\right)dx+\left(2y\sin x-x^3+\ln y\right)dy=0 \;;\; y(0)=e$$

Criteria for exactness:

$$\begin{aligned} M &= y^2\cos x - 3x^2y - 2x \;\; ; \qquad \tfrac{\partial M}{\partial y} = 2y\cos x - 3x^2 \\ N &= 2y\sin x - x^3 + \ln y \;\; ; \qquad \tfrac{\partial N}{\partial x} = 2y\cos x - 3x^2 \end{aligned}$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and the DE is exact..
Solution:

$$\begin{aligned} \tfrac{\partial f}{\partial x} &= y^2\cos x - 3x^2y - 2x \quad \Rightarrow \quad f = y^2\sin x - x^3y - x^2 + g(y) \\ \tfrac{\partial f}{\partial y} &= 2y\sin x - x^3 + \ln y \qquad \Rightarrow \quad f = y^2\sin x - x^3y + y\ln y - y + h(x) \end{aligned}$$

Thus $g(y) = y\ln y - y$; $h(x) = -x^2$, and

$$f = y^2\sin x = x^3y - x^2 + y\ln y - y$$

The general solution is

$$y^2\sin x - x^3y - x^2 + y\ln y - y = c.$$

Initial-value: $x = 0 \;\; ; \;\; y = e$

$$0 - 0 - 0 + e - e = c \Rightarrow \;\; c = 0$$

Solution of the IVP:

$$y^2\sin x - x^3y - x^2 + y\ln y - y = 0$$

14. The equation is

$$\left(2y - \frac{1}{x} + \cos 3x\right)dy + \left(\frac{y}{x^2} - 4x^3 + 3y\sin 3x\right)dx = 0$$

where we identify

$$\begin{aligned} M(x,y) &\equiv \frac{y}{x^2} - 4x^3 + 3y\sin 3x \\ N(x,y) &\equiv 2y - \frac{1}{x} + \cos 3x. \end{aligned}$$

We test for exactness by calculating the partial derivatives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{1}{x^2} + 3\sin 3x \\ \frac{\partial N}{\partial x} &= \frac{1}{x^2} - 3\sin 3x.\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the equation is not exact.

15. The equation is

$$\left(\frac{3y^2 - t^2}{y^5}\right) dy + \frac{t}{2y^4} dt = 0.$$

with $N(t, y) = \left(\frac{3y^2 - t^2}{y^5}\right)$ and $M(t, y) = \frac{t}{2y^4}$. Partial differentiation yields

$$\begin{aligned}\frac{\partial M}{\partial y} &= -\frac{2t}{y^5} \\ \frac{\partial N}{\partial t} &= -\frac{2t}{y^5},\end{aligned}$$

so that the equation is exact. Integration yields $f(t, y)$

$$\begin{aligned}f(t, y) &= \int \frac{t}{2y^4} dt + g(y) \\ &= \frac{t^2}{4y^4} + g(y) \\ f(t, y) &= \int \frac{3y^2 - t^2}{y^5} dy + h(t) \\ &= -\frac{3}{2y^2} + \frac{t^2}{4y^4} + h(t)\end{aligned}$$

Equating and grouping terms in t and y yields

$$\begin{aligned}\frac{t^2}{4y^4} + g(y) &= -\frac{3}{2y^2} + \frac{t^2}{4y^4} + h(t) \\ \Rightarrow g(y) &= -\frac{3}{2y^2} + h(t)\end{aligned}$$

i.e. $g(y) = -\frac{3}{2y^2}$ and $h(t) = 0$. Thus the solution is

$$-\frac{3}{2y^2} + \frac{t^2}{4y^4} = c.$$

Inserting the initial value $x = 1, y = 1$ yields

$$-\frac{3}{2} + \frac{1}{4} = c.$$

Finally, the solution to the initial value problem is given by

$$-\frac{1}{y^2} + \frac{t^2}{4y^4} = -\frac{5}{4}.$$

16. Setting

$$\begin{aligned} M(x,y) &\equiv x^2 + 2xy - y^2 \\ N(x,y) &\equiv y^2 + 2xy - x^2 \end{aligned}$$

we find

$$\begin{aligned} \frac{\partial M}{\partial y} &= 2x - 2y \\ \frac{\partial N}{\partial x} &= 2y - 2x. \end{aligned}$$

Obviously the equation is not exact. The given integrating factor is $\mu(x,y) = (x+y)^{-2}$. First we rewrite the differential equation as

$$[(x+y)^2 - 2y^2]dx + [(x+y)^2 - 2x^2]dy = 0.$$

Multiplying by the integrating factor yields

$$\left[1 - \frac{2y^2}{(x+y)^2}\right] dx + \left[1 - \frac{2x^2}{(x+y)^2}\right] dy = 0.$$

Using $M_1(x,y) \equiv 1 - 2y^2/(x+y)^2$ and $N_1(x,y) \equiv 1 - 2x^2/(x+y)^2$ (note the symmetry in the definitions for M_1 and N_1) we find the partial derivatives

$$\begin{aligned} \frac{\partial M_1}{\partial y} &= -\frac{4y}{(x+y)^2} + \frac{4y^2(x+y)}{(x+y)^4} \\ &= -\frac{4xy}{(x+y)^3} \\ \frac{\partial N_1}{\partial x} &= -\frac{4x}{(x+y)^2} + \frac{4x^2(x+y)}{(x+y)^4} \\ &= -\frac{4xy}{(x+y)^3} \end{aligned}$$

Thus the new equation is exact. Integration yields

$$\begin{aligned}
f(x,y) &= \int \left[1 - \frac{2y^2}{(x+y)^2}\right] dx + g(y) \\
&= x + \frac{2y^2}{x+y} + g(y) \\
&= x + \frac{2(y-x)(y+x) + 2x^2}{x+y} + g(y) \\
&= 2y - x + \frac{2x^2}{x+y} + g(y) \\
f(x,y) &= \int \left[1 - \frac{2x^2}{(x+y)^2}\right] dy + h(x) \\
&= y + \frac{2x^2}{x+y} + h(x).
\end{aligned}$$

and so

$$\begin{aligned}
2y - x + \frac{2x^2}{x+y} + g(y) &= y + \frac{2x^2}{x+y} + h(x) \\
\Rightarrow g(y) - x &= -y + h(x)
\end{aligned}$$

which suggests $h(x) = -x$ and $g(y) = -y$. Thus the solution is

$$y - x + \frac{2x^2}{x+y} = c.$$

17.

$$(2x+4)\,dx + (3y-8)\,dy = 0 \Rightarrow M\,(x,y) = 2x+4 \;\text{ and }\; N\,(x,y) = 3y-8$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \Rightarrow \text{ exact.}$$

$$f(x,y) = \int M dx + g(y) = x^2 + 4x + g(y)$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= g'\,(y) = N(x,y) = 3y - 8 \\
&\Rightarrow g'(y) = 3y - 8 \Rightarrow g(y) = \frac{3y^2}{2} - 8y + C \\
&\Rightarrow f(x,y) = x^2 + 4x + \frac{3y^2}{2} - 8y + C
\end{aligned}$$

$$\text{Solution:} \quad x^2 + 4x + \frac{3y^2}{2} - 8y + C = 0$$

18.

$$\left(x^3 + y^3\right) dx + 3xy^2 dy = 0 \Rightarrow M(x, y) = x^3 + y^3 \quad \text{and} \quad N(x, y) = 3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 = \frac{\partial N}{\partial x} \Rightarrow \text{ exact.}$$

$$f(x, y) = \int M dx + g(y) = \frac{x^4}{4} + xy^3 + g(y)$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= 3xy^2 + g'(y) = N(x, y) = 3xy^2 \\ &\Rightarrow g'(y) = 0 \Rightarrow g(y) = C \\ &\Rightarrow f(x, y) = \frac{x^4}{4} + xy^3 + C \end{aligned}$$

$$\text{Solution:} \quad \frac{x^4}{4} + xy^3 + C = 0$$

19.

$$(4y + 2t - 5)\, dt + (6y + 4t - 1)\, dy = 0 \quad y(-1) = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} = M(t, y) = 4y + 2t - 5 \quad \text{and} \quad \frac{\partial f}{\partial y} = N(t, y) = 6y + 4t - 1$$

$$\frac{\partial M}{\partial y} = 4 = \frac{\partial N}{\partial t} \Rightarrow \text{ DE is exact}$$

$$\begin{aligned} \text{Hence, } f(t, y) &= \int M dt + g(y) = 4ty + t^2 - 5t + g(y) \\ \frac{\partial f}{\partial y} &= 4t + g'(y) = N(t, y) = 6y + 4t - 1 \\ &\Rightarrow g'(y) = 6y - 1 \\ &\Rightarrow g(y) = 3y^2 - y \end{aligned}$$

$$\text{Hence, } f(t, y) = 4ty + t^2 - 5t + 3y^2 - y = C, \quad C \in \mathbb{R}$$

$$\begin{aligned} &\text{Initial value:} \quad && f(-1, 0) = 0 + 1 + 5 + 0 - 0 = 6 = C \\ &\text{and so } f(t, y) &&= 4ty + t^2 - 5t + 3y^2 - y = 6 \end{aligned}$$

20.

$$\left(y^3 + kxy^4 - 2x\right) dx + \left(3xy^2 + 20x^2y^3\right) dy = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = M(x, y) = y^3 + kxy^4 - 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = N(x, y) = 3xy^2 + 20x^2y^3$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 3y^2 + 4kxy^3 \\ \frac{\partial N}{\partial x} &= 3y^2 + 40xy^3 \\ &\Rightarrow k = 10 \text{ for the DE to be exact.} \end{aligned}$$

$$\begin{aligned} \text{Now, } f(x, y) &= \int M dx + g(y) = xy^3 + 5x^2y^4 - x^2 + g(y) \\ \frac{\partial f}{\partial y} &= 3xy^2 + 20x^2y^3 + g'(y) = N(x, y) = 3xy^2 + 20x^2y^3 \\ &\Rightarrow g'(y) = 0 \\ &\Rightarrow g(y) = k, \quad k \in \mathbb{R} \end{aligned}$$

$$\text{Hence, } f(x, y) = xy^3 + 5x^2y^4 - x^2 = C, \quad C \in \mathbb{R}$$

21.

$$\frac{dy}{dx} + P(x)y = f(x) \Rightarrow (P(x)y - f(x))dx + dy = 0$$

$$\text{Choose } M(x, y) = P(x)y - f(x) \text{ and } N(x, y) = 1$$

$$\text{Then } \frac{\partial M}{\partial y} = P(x) \neq \frac{\partial N}{\partial x} = 0.$$

$$\text{However, } e^{\int P dx}(P(x)y - f(x))dx + e^{\int P dx}dy = 0$$

$$\Rightarrow M(x, y) = e^{\int P dx}(P(x)y - f(x)) \text{ and } N(x, y) = e^{\int P dx}$$

$$\text{Now } \frac{\partial M}{\partial y} = P(x)e^{\int P dx} = \frac{\partial N}{\partial x} = P(x)e^{\int P dx}, \text{ and so the DE is exact.}$$