TWK2A
Chapter 1
Solutions

1. The first order differential equation is linear in x and nonlinear in y:

$$(y^2-1)\frac{dx}{dy}+x=0, \qquad x\frac{dy}{dx}+y^2-1=0.$$

2. 3rd order linear.

3. 2nd order nonlinear.

4.

$$\begin{aligned} y' &= \sin x \ln(\sec x+\tan x)-\cos x\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x} \\ &= \sin x\ln(\sec x+\tan x)-1 \\ y'' &= \cos x\ln(\sec x+\tan x)+\tan x \end{aligned}$$

5. (a)

$$\begin{aligned} x\phi_1'(x)-2\phi_1(x) &= 2x^2-2x^2=0, \qquad \forall x\in(-\infty,\infty) \\ x\phi_2'(x)-2\phi_2(x) &= -2x^2+2x^2=0, \qquad \forall x\in(-\infty,\infty) \end{aligned}$$

We already know that $-x^2$ is a solution on $(-\infty,0)$ and x^2 is a solution on $(0,\infty)$. Thus all that is required is to show that the equation makes sense (is defined) at $x=0$. The equation is $xy'-2y=0$. The function $y(x)$ is defined at $x=0$ as $y(0)=0$. We must determine if y' is defined at $x=0$, thus y must be continuous at $x=0$.

$$\lim_{x\to 0^-} y(x)=\lim_{x\to 0^+} y(x)=y(0)=0$$

Furthermore, we find from the definition of the derivative that $y'(0)$ exists and is zero.

$$\begin{aligned} \lim_{h\to 0^-}\frac{y(0+h)-y(0)}{h} &= \lim_{h\to 0^-}(-h)=0 \\ \lim_{h\to 0^+}\frac{y(0+h)-y(0)}{h} &= \lim_{h\to 0^+} h=0 \\ \Rightarrow y'(0) &= 0 \end{aligned}$$

The continuity of y' at 0 cannot be determined before $y'(0)$ has been determined and is thus not relevant to the question.

6.

$$\begin{aligned}
\frac{dx}{dt} &= -2e^{-2t} + 18e^{6t} \\
&= (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) \\
&= x + 3y, \qquad \forall t \in (-\infty, \infty) \\
\frac{dx}{dt} &= 2e^{-2t} + 30e^{6t} \\
&= 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) \\
&= 5x + 3y, \qquad \forall t \in (-\infty, \infty)
\end{aligned}$$

7. Order 3; highest order derivative is y'''. Nonlinear due to the term $(y')^4$.

8. The ODE is first-order. Writing it in the form $u\frac{dv}{du} + (1+u)v = ue^u$,we see that it is linear in v. However, writing it in the form $(v+uv-ue^u)\frac{du}{dv}+u = 0$, we see that it is nonlinear in u.

9. Clearly 2nd-order. Nonlinear since the RHS is nonlinear in R.

10. We have

$$\begin{aligned}
y &= e^{3x}\cos 2x \\
y' &= 3e^{3x}\cos 2x - 2e^{3x}\sin 2x \\
y'' &= 9e^{3x}\cos 2x - 6e^{3x}\sin 2x - 6e^{3x}\sin 2x - 4e^{3x}\cos 2x \\
&= 5e^{3x}\cos 2x - 12e^{3x}\sin 2x
\end{aligned}$$

Hence,

$$\begin{aligned}
y'' &- 6y' + 13y \\
&= 5e^{3x}\cos 2x - 12e^{3x}\sin 2x - 6\left(3e^{3x}\cos 2x - 2e^{3x}\sin 2x\right) + 13e^{3x}\cos 2x \\
&= e^{3x}\left[(5-18+13)\cos 2x + (-12+12)\sin 2x\right] \\
&= 0
\end{aligned}$$

and the equation is satisfied.

11. Differentiate with respect to x :

$$1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0 \Rightarrow (1 + xe^{xy})\,\frac{dy}{dx} + 1 + ye^{xy} = 0$$

Hence, the equation is reduced to an identity.

12. Order 4; highest order derivative is $y^{(4)}$. Linear.

13. The ODE is of second order since the highest order derivative is $\frac{d^2u}{dr^2}$. The equation is nonlinear in u because of the term $\cos(r + u)$.

14. The given y is defined on $(-\infty, \infty)$. Also, $\lim_{x\to 0^+} y(x) = \lim_{x\to 0^-} y(x) = 0$, i.e. $y(x)$ is continuous on $(-\infty, \infty)$. We find that

$$\frac{dy}{dx} = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

where we include 0 in the derivative since $\lim_{x\to 0^+}(-2x) = \lim_{x\to 0^-}(2x) = 0$. Thus y' is also continuous on $(-\infty, \infty)$. Finally

$$xy' - 2y = \begin{cases} x(-2x) - 2(-x^2), & x < 0 \\ x(2x) - 2(x^2), & x \geq 0 \end{cases} = \begin{cases} 0, & x < 0 \\ 0, & x \geq 0 \end{cases} = 0.$$

Consequently $y(x)$ is a solution on $(-\infty, \infty)$.

15. Differentiating the given equation on the left and right yields

$$\begin{aligned} \frac{d}{dx}(y^2 - 2y) &= \frac{d}{dx}(x^2 - x + c) \\ 2y\frac{dy}{dx} - 2\frac{dy}{dx} &= 2x - 1 \\ (2y - 2)\frac{dy}{dx} &= 2x - 1. \end{aligned}$$

Thus we have derived the given differential equation, so the equation describes an implicit solution.
Substituting $y = 1$ and $x = 0$ into the equation yields

$$1 - 2 = 0 - 0 + c,$$

i.e. $c = -1$. The implicit solution which satisfies the differential equation is then

$$y^2 - 2y = x^2 - x - 1.$$

16. There are no restrictions on the domain of y from the differential equation. However, the term $\ln x$ in $y(x)$ requires that $x > 0$. Provided $y(x)$ is a solution, the interval of definition is $(0, \infty)$ or any subinterval. The function y and its derivatives below are continuous on this interval.

$$\begin{aligned} y &= \sin(\ln x) \\ \frac{dy}{dx} &= \cos(\ln x)\frac{1}{x} \\ \frac{d^2y}{dx^2} &= -\sin(\ln x)\frac{1}{x^2} - \cos(\ln x)\frac{1}{x^2} = -\frac{1}{x^2}(\sin(\ln x) + \cos(\ln x)). \end{aligned}$$

Thus we find

$$\begin{aligned} x^2y'' + xy' + y &= x^2\left(-\frac{1}{x^2}(\sin(\ln x) + \cos(\ln x))\right) + x\left(\cos(\ln x)\frac{1}{x}\right) + \sin(\ln x) \\ &= -\sin(\ln x) - \cos(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &= 0, \end{aligned}$$

i.e. $y(x)$ is a solution on $(0, \infty)$.

17. Order 2. Nonlinear due to the term $\sqrt{y'}$.

18. The equation is non-linear in y due to the coefficient $2y$ of $\frac{dy}{dx}$. Rewriting the equation as

$$2y + 2x\frac{dx}{dy} = 0$$

we see that the equation is also non-linear in x. However the structure of the equation and the fact that $\frac{d}{dx}x^2 = 2x$ leads us to the set of equations

$$u(x, y(x)) = x^2 + y^2$$

$$\frac{du}{dx} = 2x + 2y\frac{dy}{dx} = 0$$

where the differential equation $\frac{du}{dx} = 0$ is first order linear. Thus we transformed a non-linear equation into a linear equation which can be solved trivially.

19. Since $y' = e^{-x^2} > 0$ for $x \in \mathbf{R}$ the function must be increasing for all $x \in \mathbf{R}$. $\lim_{x\to-\infty} y' = \lim_{x\to-\infty} e^{-x^2} = \lim_{x\to\infty} y' = \lim_{x\to\infty} e^{-x^2} = 0$.Consequently, as $x \to \pm\infty$ the function value $y(x)$ changes very little (and not at all in the limit). Thus $y(x)$ tends asymptotically to a constant value for $x \to \pm\infty$.

20. Since $y(x)$ is a constant we have $\frac{dy}{dx} = 0$ and $0 = y(a - by)$, so that $y_1 = 0$ and $y_2 = \frac{a}{b}$ are two constant solutions.

21. Since $y(x)$ is a constant we have $\frac{dy}{dx} = y' = 0$ and $0 = y^2 + 4$, so that $y = \pm\sqrt{-4}$. However $y \in \mathbf{R}$ and $\pm\sqrt{-4} \notin \mathbf{R}$; consequently no constant real solution exists.

22. We have

$$\begin{aligned} y &= c_1x^{-1} + c_2x + c_3x\ln x + 4x^2 \\ \frac{dy}{dx} &= -c_1x^{-2} + c_2 + c_3\ln x + c_3 + 8x \\ \frac{d^2y}{dx^2} &= 2c_1x^{-3} + c_3x^{-1} + 8 \\ \frac{d^3y}{dx^3} &= -6c_1x^{-4} - c_3x^{-2}. \end{aligned}$$

Inserting into the differential equation yields

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 12x^2.$$

Thus $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ is a solution if an appropriate interval is chosen, e.g. $x > 0$.

23. Observe that $y(x)$ is not continuous at $x = 0$; consequently the derivative $\frac{dy}{dx}$ is not defined at $x = 0$ and so the equation is not defined for all $x \in (-5, 5)$.

24. Using $y = x^m$, $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$ yields

$$x^2y'' - 7xy' + 15y = m(m-1)x^m - 7mx^m + 15x^m = (m^2 - 8m + 15)x^m = 0.$$

Obviously the equation is satisfied for $x = 0$; assuming $x \neq 0$ leads to the equation

$$m^2 - 8m + 15 = (m-3)(m-5) = 0$$

so that the two solutions are given by $m_1 = 3$ and $m_2 = 5$, and consequently $y_1 = x^3$ and $y_2 = x^5$ are the required solutions.

25. Using $y(x) = c_1 e^x + c_2 e^{-x}$ and $y'(x) = c_1 e^x - c_2 e^{-x}$ we find the conditions

$$\begin{aligned} y(1) &= c_1 e + \frac{c_2}{e} = 0 \\ y'(1) &= c_1 e - \frac{c_2}{e} = e \end{aligned}$$

Inserting $c_2 = -c_1 e^2$ from the first equation into the second equation yields

$$c_1 e + c_1 e = e$$

so that $c_1 = \frac{1}{2}$ and $c_2 = -\frac{e^2}{2}$. Thus the solution is

$$y = \frac{1}{2} e^x - \frac{1}{2} e^{-x+2}.$$

26. Using $y(x) = c_1 e^x + c_2 e^{-x}$ and $y'(x) = c_1 e^x - c_2 e^{-x}$ we find the conditions

$$\begin{aligned} y(0) &= c_1 + c_2 = 0 \\ y'(0) &= c_1 - c_2 = 0 \end{aligned}$$

Inserting $c_2 = -c_1$ from the first equation into the second equation yields $2c_1 = 0$, so that $c_1 = 0$ and $c_2 = 0$. Thus the solution is $y(x) = 0$.

27. Differentiating yields

$$\frac{d}{dx} \cos(x - y) = \frac{d}{dx} c$$

so that

$$-\sin(x - y) \left(1 - \frac{dy}{dx}\right) = 0.$$

Differentiating again

$$\frac{d^2}{dx^2} \cos(x - y) = \frac{d^2}{dx^2} c$$

and substituting the first derivative determined above

$$\frac{d}{dx} \left[-\sin(x - y) \left(1 - \frac{dy}{dx}\right) \right] = \frac{d}{dx} 0$$

gives

$$-\cos(x - y) \left(1 - \frac{dy}{dx}\right)^2 + \sin(x - y) \frac{d^2 y}{dx^2} = 0$$

which yields the differential equation (after rearranging and division by $\cos(x-y)$ since $\cos(x-y) \neq 0$ from the DE)

$$\tan(x-y)\frac{d^2y}{dx^2} = \left(1 - \frac{dy}{dx}\right)^2.$$

The initial value $y(1) = 1$ yields $\cos(1-1) = 1 = c$. Thus the solution is $\cos(x-y) = 1$.

Note 1: It may be tempting to rewrite the given implicit solution as

$$x - y = \arccos(c)$$

and solve for y, and then differentiate twice. However the range of arccos is $[0, \pi]$ and consequently this method imposes the additional constraint $x - y \in [0, \pi]$. Thus the method given above is more general since it does not impose this constraint.

Note 2: We can solve directly for $\frac{dy}{dx}$ to find $\frac{dy}{dx} = 1$ from

$$-\sin(x-y)\left(1 - \frac{dy}{dx}\right) = 0$$

only when $\sin(x-y) \neq 0$. This imposes an additional constraint on x and y. The solution given above does not suffer this constraint and is more general.