TWK2A
Additional operational properties of the Laplace transform (Section 7.4)
Solutions

1.

$$\begin{aligned}
\mathcal{L}\{y' - y\} &= \mathcal{L}\left\{te^t \sin t\right\} \\
s\mathcal{L}\{y\} - y(0) - \mathcal{L}\{y\} &= (-1)\frac{d}{ds}\mathcal{L}\left\{e^t \sin t\right\} \\
(s-1)\mathcal{L}\{y\} &= (-1)\frac{d}{ds}\left[\mathcal{L}\{\sin t\}|_{s\to s-1}\right] \\
&= -\frac{d}{ds}\frac{1}{(s-1)^2+1} \\
&= \frac{2(s-1)}{((s-1)^2+1)^2} \\
\mathcal{L}\{y\} &= \frac{2}{((s-1)^2+1)^2} \\
y &= e^t\mathcal{L}^{-1}\left\{\frac{2}{(s^2+1)^2}\right\} \\
&= e^t(\sin t - t\cos t).
\end{aligned}$$

2.

$$\begin{aligned}
\mathcal{L}\{y'' + 9y\} &= \mathcal{L}\{\cos 3t\} \\
s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 9\mathcal{L}\{y\} &= \frac{s}{s^2+9} \\
(s^2+9)\mathcal{L}\{y\} &= \frac{s}{s^2+9} + 2s + 5 \\
\mathcal{L}\{y\} &= \frac{s}{(s^2+9)^2} + 2\frac{s}{s^2+9} + 5\frac{1}{s^2+9} \\
y &= \frac{1}{6}t\sin 3t + 2\cos 3t + \frac{5}{3}\sin 3t.
\end{aligned}$$

3.

$$
\begin{aligned}
\mathcal{L}\{y''+y\} &= \mathcal{L}\left\{1-\mathcal{U}\left(t-\frac{\pi}{2}\right)+(\sin t)\,\mathcal{U}\left(t-\frac{\pi}{2}\right)\right\} \\
s^2\mathcal{L}\{y\}-sy(0)-y'(0)+\mathcal{L}\{y\} &= \frac{1}{s}-\frac{e^{-\frac{\pi}{2}s}}{s}+\mathcal{L}\left\{\underbrace{\sin\left(t+\frac{\pi}{2}\right)}_{\cos t}\mathcal{U}\left(t-\frac{\pi}{2}\right)\right\} \\
(s^2+1)\mathcal{L}\{y\} &= \frac{1}{s}-\frac{e^{-\frac{\pi}{2}s}}{s}+\left(\frac{s}{s^2+1}\right)e^{-\frac{\pi}{2}s}+s \\
&= \frac{1+s^2}{s}+e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}-\frac{1}{s}\right) \\
\mathcal{L}\{y\} &= \frac{1}{s}+e^{-\frac{\pi}{2}s}\left(\frac{s}{(s^2+1)^2}-\frac{1}{s(s^2+1)}\right) \\
y &= 1+\left[\frac{1}{2}t\sin t-2\sin^2\frac{t}{2}\right]_{t\to t-\frac{\pi}{2}}\mathcal{U}\left(t-\frac{\pi}{2}\right) \\
&= 1+\left[\frac{1}{2}t\sin t+\cos t-1\right]_{t\to t-\frac{\pi}{2}}\mathcal{U}\left(t-\frac{\pi}{2}\right) \\
&= 1+\left[\frac{1}{2}\left(t-\frac{\pi}{2}\right)\sin\left(t-\frac{\pi}{2}\right)+\cos\left(t-\frac{\pi}{2}\right)-1\right]\cdot \\
&\quad\ \mathcal{U}\left(t-\frac{\pi}{2}\right) \\
&= 1+\left[-\frac{1}{2}\left(t-\frac{\pi}{2}\right)\cos t+\sin t-1\right]\mathcal{U}\left(t-\frac{\pi}{2}\right) \\
&= 1+\left[-\frac{1}{2}t\cos t+\frac{\pi}{4}\cos t+\sin t-1\right]\mathcal{U}\left(t-\frac{\pi}{2}\right) \\
&= \begin{cases} 1, & 0\leqslant t<\frac{\pi}{2} \\ -\frac{1}{2}t\cos t+\frac{\pi}{4}\cos t+\sin t & t\geqslant\frac{\pi}{2} \end{cases}
\end{aligned}
$$

Note that the term $\frac{\pi}{4}\cos t+\sin t$ is a solution to the corresponding homogeneous differential equation. This term ensures the continuity of y, y' and y'' at $\frac{\pi}{2}$.

4.

$$\mathcal{L}\left\{te^{2t}\sin 6t\right\}=(-1)\frac{dF(s)}{ds}$$

where

$$\begin{aligned} F(s) &= \mathcal{L}\left\{e^{2t}\sin 6t\right\} \\ &= \frac{6}{(s-2)^2+36} \\ &= \frac{6}{s^2-4s+40}. \end{aligned}$$

Hence,

$$\mathcal{L}\left\{te^{2t}\sin 6t\right\} = \frac{12s-24}{\left(s^2-4s+40\right)^2}.$$

5. We have

$$f(t) = \begin{cases} 1, & 0 \leqslant t < a \\ -1, & a \leqslant t < 2a \end{cases}$$

We see that $T = 2a$. Hence,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-2as}}\left(\int_0^a e^{-st}dt + \int_a^{2a} -e^{-st}dt\right) \\ &= \frac{1}{1-e^{-2as}}\left(\frac{1-e^{-as}}{s} + \frac{e^{-2as}-e^{-as}}{s}\right) \\ &= \frac{1}{1-e^{-2as}}\left(\frac{1-2e^{-as}+e^{-2as}}{s}\right) \\ &= \frac{\left(1-e^{-as}\right)^2}{s\left(1-e^{-2as}\right)} \\ &= \frac{\tanh\frac{as}{2}}{s}. \end{aligned}$$

6. Here, $T = b$ and $f(t) = \frac{at}{b}$ on $[0, b]$. Hence,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{1-e^{-bs}} \int_0^b \frac{at}{b} e^{-st} dt = \frac{a}{b}\left(\frac{1}{1-e^{-bs}}\right) \int_0^b t e^{-st} dt \\
&= \frac{a}{b}\left(\frac{1}{1-e^{-bs}}\right)\left[\frac{-ste^{-st} - e^{-st}}{s^2}\right]_0^b \\
&= \frac{a}{b}\left(\frac{1}{1-e^{-bs}}\right)\left(\frac{1-e^{-bs} - bse^{-bs}}{s^2}\right) \\
&= \left(\frac{a}{s}\right)\left(\frac{1-e^{-bs} - bse^{-bs}}{bs\left(1-e^{-bs}\right)}\right) \\
&= \left(\frac{a}{s}\right)\left(\frac{1}{bs} + \frac{-e^{-bs}}{1-e^{-bs}}\right) \\
&= \left(\frac{a}{s}\right)\left(\frac{1}{bs} + \frac{1}{1-e^{bs}}\right).
\end{aligned}$$