TWK2A
Second translation theorem (Section 7.3.2)
Solutions

1.

$$\mathcal{L}\left\{e^{2-t}\,\mathcal{U}(t-2)\right\} = \mathcal{L}\left\{e^{-(t-2)}\,\mathcal{U}(t-2)\right\}$$

From the Second Translation Theorem we have

$$\begin{aligned} a &= 2 \\ f(t) &= e^{-t} \;\; ; \;\; f(t-a) = f(t-2) = e^{-(t-2)}. \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}\left\{e^{2-t}\,\mathcal{U}(t-2)\right\} &= e^{-2s}\mathcal{L}\left\{e^{-t}\right\} \\ &= e^{-2s}\left(\frac{1}{s+1}\right). \end{aligned}$$

2.

$$\mathcal{L}\left\{\sin t\,\mathcal{U}\left(t-\frac{\pi}{2}\right)\right\}$$

Using the alternative form of the Second Translation Theorem we have

$$\begin{aligned} a = \frac{\pi}{2}, \quad g(t) = \sin t, \quad g(t+a) = g\left(t+\frac{\pi}{2}\right) &= \sin\left(t+\frac{\pi}{2}\right) \\ &= \cos t. \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}\left\{\sin t\,\mathcal{U}\left(t-\frac{\pi}{2}\right)\right\} &= e^{-\frac{\pi}{2}s}\mathcal{L}\left\{\cos t\right\} \\ &= e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+1}\right). \end{aligned}$$

3.

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$$

$$a = \pi$$

$$F(s) = \frac{1}{s^2+1} \Rightarrow f(t) = \sin t.$$

So

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} &= f(t-\pi)\,\mathcal{U}(t-\pi) \\
&= \sin(t-\pi)\,\mathcal{U}(t-\pi) \\
&= -\sin(t)\,\mathcal{U}(t-\pi).
\end{aligned}$$

4.

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\}$$

Partial Fractions:

$$\begin{aligned}
\frac{1}{s^2(s-1)} &= \frac{As+B}{s^2} + \frac{C}{s-1} \\
&= \frac{As^2 - As + Bs - B + Cs^2}{s^2(s-1)} \\
&= \frac{(A+C)s^2 + (-A+B)s - B}{s^2(s-1)}
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow\ A + C = 0 \quad -A + B = 0 \quad -B = 1 \\
&\Rightarrow\ A = -1, \quad C = 1, \quad B = -1 \\
&\Rightarrow\ \frac{1}{s^2(s-1)} = \frac{-s-1}{s^2} + \frac{1}{s-1}
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\} &= \mathcal{L}^{-1}\left\{-\frac{e^{-2s}s}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s-1}\right\} \\
&= -\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}\right\} \\
&= -\,\mathcal{U}(t-2) - (t-2)\ \mathcal{U}(t-2) + e^{(t-2)}\ \mathcal{U}(t-2) \\
&= \left(1 - t + e^{(t-2)}\right)\mathcal{U}(t-2).
\end{aligned}$$

$$\left(\begin{array}{lcrclcr} \text{For the first term } F(s) & = & \frac{1}{s} & \text{so} & f(t) & = & 1 \\ \text{For the second term } F(s) & = & \frac{1}{s^2} & \text{so} & f(t) & = & t \\ \text{For the third term } F(s) & = & \frac{1}{s-1} & \text{so} & f(t) & = & e^t \end{array}\right)$$

5.

$$f(t) = \begin{cases} 1, & 0 \leqslant t < 4 \\ 0, & 4 \leqslant t < 5 \\ 1, & t \geqslant 5 \end{cases}$$

So

$$\begin{aligned} f(t) &= 1 - \mathcal{U}(t-4) + \mathcal{U}(t-5) \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{1) - \mathcal{L}\{\mathcal{U}(t-4) + \mathcal{L}\left\{\mathcal{U}(t-5)\right\} \\ &= \frac{1}{s} - \frac{e^{-4s}}{s} + \frac{e^{-5s}}{s}. \end{aligned}$$

6.

$$\begin{aligned} \mathcal{L}\left\{(t-1)\,\mathcal{U}\,(t-1)\right\} &= e^{-s}\mathcal{L}\left\{t\right\} \\ &= \frac{e^{-s}}{s^2}. \end{aligned}$$

7.

$$\begin{aligned} \mathcal{L}\left\{\cos 2t\;\mathcal{U}\,(t-\pi)\right\} &= e^{-\pi s}\mathcal{L}\left\{\cos 2\,(t+\pi)\right\} \\ &= e^{-\pi s}\mathcal{L}\left\{\cos\,(2t+2\pi)\right\} \\ &= e^{-\pi s}\mathcal{L}\left\{\cos 2t\right\} \\ &= \frac{se^{-\pi s}}{s^2+4}. \end{aligned}$$

8.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} &= \mathcal{U}(t-1)\left(\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}\right)_{t\to t-1} \\
&= \mathcal{U}(t-1)\frac{(t-1)^2}{2} \\
&= \mathcal{U}(t-1)\left(\frac{t^2-2t+1}{2}\right) \\
&= \frac{t^2}{2}\mathcal{U}(t-1) - t\mathcal{U}(t-1) + \frac{1}{2}\mathcal{U}(t-1).
\end{aligned}$$

9.

$$\begin{aligned}
f(t) &= \sin t\,(1-\mathcal{U}(t-2\pi)) + t^2\mathcal{U}(t-3\pi) \\
&= \sin t - \sin t\,\mathcal{U}(t-2\pi) + t^2\mathcal{U}(t-3\pi)
\end{aligned}$$

and so

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\left\{\sin t - \sin t\,\mathcal{U}(t-2\pi) + t^2\mathcal{U}(t-3\pi)\right\} \\
&= \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\} + e^{-3\pi s}\mathcal{L}\left\{(t+3\pi)^2\right\} \\
&= \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin t\} + e^{-3\pi s}\mathcal{L}\left\{t^2+6\pi t+9\pi^2\right\} \\
&= \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} + e^{-3\pi s}\left(\frac{2}{s^3}+\frac{6\pi}{s^2}+\frac{9\pi^2}{s}\right) \\
&= \frac{1-e^{-2\pi s}}{s^2+1} + e^{-3\pi s}\left(\frac{2}{s^3}+\frac{6\pi}{s^2}+\frac{9\pi^2}{s}\right).
\end{aligned}$$

10.

$$\begin{aligned}
y'+2y &= \begin{cases} t, & 0\leqslant t<1 \\ 0, & t\geqslant 1 \end{cases} \qquad y(0)=0 \\
&= t - t\,\mathcal{U}(t-1) \\
\mathcal{L}\{y'\}+2\mathcal{L}\{y\} &= \mathcal{L}\{t\} - \mathcal{L}\{t\mathcal{U}(t-1)\} \\
\Rightarrow sY - y(0) + 2Y &= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} \\
&= \frac{1}{s^2} - e^{-s}\mathcal{L}\{t\} - e^{-s}\mathcal{L}\{1\} \\
&= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}.
\end{aligned}$$

So

$$
\begin{aligned}
Y(s+2) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\
Y &= \frac{1}{s^2(s+2)} - \frac{e^{-s}}{s^2(s+2)} - \frac{e^{-s}}{s(s+2)}.
\end{aligned}
$$

Partial fractions:

$$
\begin{aligned}
\frac{1}{s^2(s+2)} &= \frac{As+B}{s^2} + \frac{C}{s+2} = \frac{As^2+Bs+2As+2B+Cs^2}{s^2(s+2)} \\
&\Rightarrow\ A+C=0, \quad B+2A=0, \quad 2B=1 \\
&\Rightarrow\ B=\frac{1}{2}, \quad A=-\frac{1}{4}, \quad C=\frac{1}{4}.
\end{aligned}
$$

So

$$
\frac{1}{s^2(s+2)} = \frac{-\frac{1}{4}s}{s^2} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2}.
$$

Also

$$
\frac{1}{s(s+2)} = \frac{1}{s^2+2s} = \frac{1}{(s+1)^2-1} = \left.\frac{1}{s^2-1}\right|_{s\to s+1}.
$$

So

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2}\right\} \\
&\quad -\mathcal{L}^{-1}\left\{\frac{-\frac{1}{4}e^{-s}}{s} + \frac{\frac{1}{2}e^{-s}}{s^2} + \frac{\frac{1}{4}e^{-s}}{s+2}\right\} \\
&\quad -\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2-1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} \\
&\quad + \frac{1}{4}\,\mathcal{U}(t-1) - \frac{1}{2}(t-1)\,\mathcal{U}(t-1) - \frac{1}{4}e^{-2(t-1)}\,\mathcal{U}(t-1) \\
&\quad \underbrace{-\,\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2-1}\right|_{s\to s+1}\right\}_{t\to t-1}\ \mathcal{U}(t-1)}_{e^{-t}\sinh t|_{t\to t-1}\ \mathcal{U}(t-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t} \\
&\quad +\frac{1}{4}\,\mathcal{U}(t-1)-\frac{1}{2}(t-1)\,\mathcal{U}(t-1)-\frac{1}{4}e^{-2(t-1)}\,\mathcal{U}(t-1) \\
&\quad -e^{-(t-1)}\sinh(t-1)\mathcal{U}(t-1) \\
&= -\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}+\left(\frac{1}{4}-\frac{1}{2}t+\frac{1}{4}e^{-2t}\right)\,\mathcal{U}(t-1).
\end{aligned}
$$

Note that, in the above, we have used

$$\sinh(t-1)=\frac{e^{t-1}-e^{-t+1}}{2}.$$

11.

$$y''+4y=1-\mathcal{U}\left(t-1\right)$$

Now,

$$\mathcal{L}\left\{y''+4y\right\}=\mathcal{L}\left\{1-\mathcal{U}\left(t-1\right)\right\}$$

$$
\begin{aligned}
&\Rightarrow\quad s^2Y\left(s\right)-sy\left(0\right)-y'\left(0\right)+4Y\left(s\right)=\frac{1}{s}-\frac{e^{-s}}{s} \\
&\Rightarrow\quad s^2Y\left(s\right)+1+4Y\left(s\right)=\frac{1}{s}-\frac{e^{-s}}{s} \\
&\Rightarrow\quad Y\left(s\right)\left(s^2+4\right)=\frac{1}{s}-\frac{e^{-s}}{s}-1 \\
&\Rightarrow\quad Y\left(s\right)=\frac{1}{s\left(s^2+4\right)}-\frac{e^{-s}}{s\left(s^2+4\right)}-\frac{1}{s^2+4}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+4\right)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s\left(s^2+4\right)}\right\} \\
&= \frac{1}{2}\sin^2 t
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s\left(s^2+4\right)}\right\} &= \mathcal{U}\left(t-1\right)\mathcal{L}^{-1}\left\{\frac{1}{s\left(s^2+4\right)}\right\}_{t\to t-1} \\
&= \mathcal{U}\left(t-1\right)\left(\frac{1}{2}\sin^2 t\right)_{t\to t-1} \\
&= \left(\frac{1}{2}\sin^2\left(t-1\right)\right)\mathcal{U}\left(t-1\right).
\end{aligned}
$$

Hence,

$$y(t) = \frac{1}{2}\sin^2 t - \left(\frac{1}{2}\sin^2(t-1)\right)\mathcal{U}(t-1) - \frac{\sin 2t}{2}.$$

12. Note that

$$\begin{aligned}
\frac{s}{(s+7)^3} &= \left.\frac{s-7}{s^3}\right|_{s\to s+7} \\
&= \left.\frac{s}{s^3}\right|_{s\to s+7} - \left.\frac{7}{s^3}\right|_{s\to s+7} \\
&= \left.\frac{1}{s^2}\right|_{s\to s+7} - \left.\frac{7}{s^3}\right|_{s\to s+7}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s+7)^3}\right\} &= \mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s+7}\right\} - \mathcal{L}^{-1}\left\{\left.\frac{7}{s^3}\right|_{s\to s+7}\right\} \\
&= \mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s+7}\right\} - \frac{7}{2}\mathcal{L}^{-1}\left\{\left.\frac{2}{s^3}\right|_{s\to s+7}\right\} \\
&= te^{-7t} - \frac{7}{2}t^2e^{-7t}.
\end{aligned}$$

13. Note that

$$\begin{aligned}
\frac{s^2}{(s-3)^4} &= \left.\frac{(s+3)^2}{s^4}\right|_{s\to s-3} = \left.\frac{s^2+6s+9}{s^4}\right|_{s\to s-3} \\
&= \left.\frac{s^2}{s^4}\right|_{s\to s-3} + \left.\frac{6s}{s^4}\right|_{s\to s-3} + \left.\frac{9}{s^4}\right|_{s\to s-3} \\
&= \left.\frac{1}{s^2}\right|_{s\to s-3} + \left.\frac{6}{s^3}\right|_{s\to s-3} + \left.\frac{9}{s^4}\right|_{s\to s-3}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s^2}{(s-3)^4}\right\} &= \mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\} + \mathcal{L}^{-1}\left\{\left.\frac{6}{s^3}\right|_{s\to s-3}\right\} + \mathcal{L}^{-1}\left\{\left.\frac{9}{s^4}\right|_{s\to s-3}\right\} \\
&= \mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s-3}\right\} + \frac{6}{2}\mathcal{L}^{-1}\left\{\left.\frac{2}{s^3}\right|_{s\to s-3}\right\} + \frac{9}{3!}\mathcal{L}^{-1}\left\{\left.\frac{3!}{s^4}\right|_{s\to s-3}\right\} \\
&= te^{3t} + 3t^2e^{3t} + \frac{3}{2}t^3e^{3t}.
\end{aligned}$$