TWK2A
First translation theorem (Section 7.3.1)
Solutions

1.

$$\mathcal{L}\left\{t^3e^{-2t}\right\}$$

Using the First Translation Theorem we have

$$\begin{aligned} a &= -2 \\ f(t) &= t^3 \end{aligned}$$

So $F(s) = \frac{6}{s^4}$; $F(s-a) = F(s+2) = \frac{6}{(s+2)^4}$.
Hence,

$$\mathcal{L}\left\{t^3e^{-2t}\right\} = \frac{6}{(s+2)^4}.$$

2.

$$\mathcal{L}\left\{e^{-2t}\cos 4t\right\}$$

Using the First Translation Theorem we have

$$\begin{aligned} a &= -2 \\ f(t) &= \cos 4t \end{aligned}$$

So

$$\begin{aligned} F(s) = \frac{s}{s^2+16} \Rightarrow F(s-a) = F(s+2) &= \frac{s+2}{(s+2)^2+16} \\ &= \frac{s+2}{s^2+4s+20}. \end{aligned}$$

3.

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$$

Partial fractions:

$$\frac{s}{s^2+4s+5} = \frac{s}{(s+2)^2+1} = \frac{s+2-2}{(s+2)^2+1}$$
$$= \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}.$$

So

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+1}\right\}$$
$$= \mathcal{L}^{-1}\left\{\left.\frac{s}{s^2+1}\right|_{s\to s+2}\right\} - 2\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2+1}\right|_{s\to s+2}\right\}$$
$$= e^{-2t}\cos t - 2e^{-2t}\sin t.$$

4.

$$\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+4)^4}\right\}$$

Assume $\frac{(s+1)^2}{(s+4)^4} = F(s+4)$ (i.e. is a function of $s+4$)
Define $w \equiv s+4$. Hence, $s+1 = w-3$ and so

$$\frac{(s+1)^2}{(s+4)^4} = \frac{(w-3)^2}{w^4} = \frac{w^2-6w+9}{w^4}$$
$$= \frac{1}{w^2} - \frac{6}{w^3} + \frac{9}{w^4}$$
$$= \frac{1}{(s+4)^2} - \frac{6}{(s+4)^3} + \frac{9}{(s+4)^4}.$$

So

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+4)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}-\frac{6}{(s+4)^3}+\frac{9}{(s+4)^4}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}-6\mathcal{L}^{-1}\left\{\frac{1}{(s^4)^3}\right\}+9\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^4}\right\} \\
&= \mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s+4}\right\}-6\mathcal{L}^{-1}\left\{\left.\frac{1}{s^3}\right|_{s\to s+4}\right\}+9\mathcal{L}^{-1}\left\{\left.\frac{1}{s^4}\right|_{s\to s+4}\right\} \\
&= e^{-4t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-6e^{4t}\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}+9e^{-4t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} \\
&= e^{-4t}t-6e^{-4t}\frac{t^2}{2}+9e^{-4t}\frac{t^3}{6} \\
&= e^{-4t}\left(t-3t^2+\frac{3t^3}{2}\right).
\end{aligned}$$

5.

$$\begin{aligned}
\mathcal{L}\left\{e^{-7t}t^{10}\right\} &= \mathcal{L}\left\{t^{10}\right\}_{s\to s+7} \\
&= \left.\frac{10!}{s^{11}}\right|_{s\to s+7} \\
&= \frac{10!}{(s+7)^{11}}.
\end{aligned}$$

6.

$$\begin{aligned}
\frac{2s-1}{(s+1)^3} &= \left.\frac{2s-3}{s^3}\right|_{s\to s+1} \\
&= 2\left(\left.\frac{1}{s^2}\right|_{s\to s+1}\right)-3\left(\left.\frac{1}{s^3}\right|_{s\to s+1}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s-1}{(s+1)^3}\right\} &= 2\mathcal{L}^{-1}\left\{\left.\frac{1}{s^2}\right|_{s\to s+1}\right\}-\frac{3}{2}\mathcal{L}^{-1}\left\{\left.\frac{2}{s^3}\right|_{s\to s+1}\right\} \\
&= 2te^{-t}-\frac{3}{2}t^2e^{-t}.
\end{aligned}$$

7.

$$\begin{aligned}
\mathcal{L}\left\{e^{2t}(t-1)^2\right\} &= \mathcal{L}\left\{t^2e^{2t}-2te^{2t}+e^{2t}\right\}\\
&= \mathcal{L}\left\{t^2e^{2t}\right\}-2\mathcal{L}\left\{te^{2t}\right\}+\mathcal{L}\left\{e^{2t}\right\}\\
&= \left.\frac{2}{s^3}\right|_{s\to s-2}-2\left.\frac{1}{s^2}\right|_{s\to s-2}+\left.\frac{1}{s}\right|_{s\to s-2}\\
&= \frac{2}{(s-2)^3}-\frac{2}{(s-2)^2}+\frac{1}{(s-2)}\\
&= \frac{s^2-6s+10}{(s-2)^3}.
\end{aligned}$$

8.

$$\begin{aligned}
\mathcal{L}\left\{e^{2t}\left(9-4t+10\sin\frac{t}{2}\right)\right\} &= 9\mathcal{L}\left\{e^{2t}\right\}-4\mathcal{L}\left\{te^{2t}\right\}+10\mathcal{L}\left\{\sin\left(\frac{t}{2}\right)e^{2t}\right\}\\
&= \frac{9}{s-2}-\frac{4}{(s-2)^2}+10\left(\left.\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\right|_{s\to s-2}\right)\\
&= \frac{9}{s-2}-\frac{4}{(s-2)^2}+\frac{5}{(s-2)^2+\frac{1}{4}}.
\end{aligned}$$

9.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\} &= \mathcal{L}^{-1}\left\{\left.\frac{1}{s^4}\right|_{s\to s-1}\right\}=\frac{1}{6}\mathcal{L}^{-1}\left\{\left.\frac{6}{s^4}\right|_{s\to s-1}\right\}\\
&= \frac{t^3e^t}{6}.
\end{aligned}$$

10.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\} &= 5\mathcal{L}^{-1}\left\{\left.\frac{s+2}{s^2}\right|_{s\to s-2}\right\}\\
&= 5\mathcal{L}^{-1}\left\{\left.\frac{s}{s^2}\right|_{s\to s-2}+\left.\frac{2}{s^2}\right|_{s\to s-2}\right\}\\
&= 5\mathcal{L}^{-1}\left\{\left.\frac{1}{s}\right|_{s\to s-2}+\left.\frac{2}{s^2}\right|_{s\to s-2}\right\}\\
&= 5\left(e^{2t}+2te^{2t}\right)\\
&= 5e^{2t}+10te^{2t}.
\end{aligned}$$