TWK2A
IVPs using the Laplace transform (Section 7.2)
Solutions

1.

$$\begin{aligned}
\mathcal{L}\left\{y' + 6y\right\} &= \mathcal{L}\left\{e^{4t}\right\} \\
s\mathcal{L}\left\{y\right\} - y(0) + 6\mathcal{L}\left\{y\right\} &= \frac{1}{s-4} \\
(s+6)\mathcal{L}\left\{y\right\} &= \frac{1}{s-4} + 2 \\
\mathcal{L}\left\{y\right\} &= \frac{1}{(s-4)(s+6)} + \frac{2}{s+6} \\
\mathcal{L}^{-1}\left\{\mathcal{L}\left\{y\right\}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-4)(s+6)} + \frac{2}{s+6}\right\} \\
y(t) &= \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} + 2e^{-6t} \\
&= \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}.
\end{aligned}$$

2.

$$\begin{aligned}
\mathcal{L}\{y'' - 4y'\} &= \mathcal{L}\left\{6e^{3t} - 3e^{-t}\right\} \\
s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 4s\mathcal{L}\{y\} + 4y(0) &= \frac{6}{s-3} - \frac{3}{s+1} \\
(s^2 - 4s)\mathcal{L}\{y\} &= \frac{6}{s-3} - \frac{3}{s+1} + s - 5 \\
\mathcal{L}\{y\} &= \frac{6}{s(s-4)(s-3)} - \frac{3}{s(s-4)(s+1)} \\
&\quad + \frac{1}{s-4} - \frac{5}{s(s-4)} \\
&= \frac{1}{2}\left(\frac{1}{s} + \frac{3}{s-4} - \frac{4}{s-3}\right) \\
&\quad + 3\left(\frac{1}{4s} - \frac{1}{20(s-4)} - \frac{1}{5(s+1)}\right) \\
&\quad + \frac{1}{s-4} + \frac{5}{4s} - \frac{5}{4(s-4)} \\
&= \frac{10}{4s} + \frac{11}{10(s-4)} - \frac{2}{s-3} - \frac{3}{5(s+1)} \\
y(t) &= \frac{10}{4} + \frac{11}{10}e^{4t} - 2e^{3t} - \frac{3}{5}e^{-t}
\end{aligned}$$

3.

$$2y''' + 3y'' - 3y' - 2y = e^{-t}; \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$$

Laplace transform:

$$\begin{aligned}
2\mathcal{L}\{y'''(t)\} + 3\mathcal{L}\{y''(t)\} &- 3\mathcal{L}\{y'(t)\} - 2\mathcal{L}\{y(t)\} = \mathcal{L}\left\{e^{-t}\right\} \\
2\left[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right] &+ 3\left[s^2Y(s) - sy(0) - y'(0)\right] \\
-3\left[sY(s) - y(0)\right] - 2Y(s) &= \frac{1}{s+1} \\
Y(s) &= \frac{2s+3}{(2s^3 + 3s^2 - 3s - 2)(s+1)} \\
&= \frac{2s+3}{(s-1)(s+1)(s+2)(2s+1)}.
\end{aligned}$$

Decompose the right hand side into partial fractions:

$$Y(s) = \frac{\frac{5}{18}}{s-1} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{9}}{s+2} - \frac{\frac{16}{9}}{2s+1}.$$

Inverse transform:

$$\begin{aligned}\mathcal{L}^{-1}\left\{Y(s)\right\} &= \frac{5}{18}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}-\frac{8}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}\\ y(t) &= \frac{5}{18}\,e^{t}+\frac{1}{2}\,e^{-t}+\frac{1}{9}\,e^{-2t}-\frac{8}{9}\,e^{-\frac{t}{2}}.\end{aligned}$$

4.

$$2y'+y=0; \qquad y(0)=-3$$

Transform of ODE:

$$\begin{aligned}2\,\mathcal{L}\{y'\}+\mathcal{L}\{y\} &= \mathcal{L}\{0\}\\ \Rightarrow 2\left[Y(s)-y(0)\right]+Y(s) &= 0\\ \Rightarrow 2sY-2(-3)+Y &= 0\\ \Rightarrow Y(s)=\frac{-6}{2s+1} &= \frac{-3}{s+\frac{1}{2}}\end{aligned}$$

Therefore

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y(s)\}=-3\ \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}\\ &= -3\,e^{-\frac{1}{2}t}.\end{aligned}$$

5.

$$y''+9y=e^{t}; \qquad y(0)=0, \qquad y'(0)=0$$

Transform of ODE:

$$\begin{aligned}\mathcal{L}\{y''\}+9\,\mathcal{L}\{y\} &= \mathcal{L}\{e^{t}\}\\ \Rightarrow s^2Y(s)-s\,y(0)-y'(0)+9\,Y(s) &= \frac{1}{s-1}\\ \Rightarrow \left(s^2+9\right)Y-0-0 &= \frac{1}{s-1}\\ \Rightarrow Y &= \frac{1}{(s-1)(s^2+9)}.\end{aligned}$$

Let

$$Y=\frac{A}{s-1}+\frac{Bs+C}{s^2+9}.$$

Then

$$\begin{array}{lll}
A(s^2+9)+(Bs+C)(s-1) & = & 1 \\
\text{Put } s=1: & & 10A=1; \qquad A=\frac{1}{10} \\
\text{Coefficient of } s^2: & & A+B=1; \qquad B=-A=-\frac{1}{10} \\
\text{Coefficient of } s^0: & & 9A-C=1; \qquad C=9A-1=-\frac{1}{10}
\end{array}$$

Hence

$$Y(s)=\frac{1}{10}\cdot\frac{1}{s-1}-\frac{1}{10}\cdot\frac{s}{s^2+9}-\frac{1}{10}\cdot\frac{1}{s^2+9}$$

and

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \frac{1}{10}\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{10}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-\frac{1}{30}\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \\
&= \frac{1}{10}\,e^t-\frac{1}{10}\,\cos 3t-\frac{1}{30}\,\sin 3t.
\end{aligned}$$

6.

$$y'''+2y''-y'-2y=\sin 3t; \qquad y(0)=0; \qquad y'(0)=0; \qquad y''(0)=1$$

Transform of ODE:

$$\begin{aligned}
\mathcal{L}\{y'''\}-2\,\mathcal{L}\{y''\}-\mathcal{L}\{y'\}-2\,\mathcal{L}\{y\} &= \mathcal{L}\{\sin 3t\} \\
\Rightarrow s^3Y(s)-s^2y(0)-sy'(0)-y''(0)+ & \\
2\left[s^2Y(s)-sy(0)-y'(0)\right]-\left[sY(s)-y(0)\right]-2Y(s) &= \frac{3}{s^2+9} \\
\Rightarrow \left(s^3+2s^2-s-2\right)Y-1 &= \frac{3}{s^2+9} \\
\Rightarrow (s-1)(s+1)(s+2)Y &= 1+\frac{3}{s^2+9}.
\end{aligned}$$

We therefore have

$$Y=\frac{s^2+12}{(s-1)(s+1)(s+2)(s^2+9)}.$$

Let

$$Y=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{Ds+E}{s^2+9}.$$

Then

$$\begin{aligned} A(s+1)(s+2)(s^2+9)+B(s-1)(s+2)(s^2+9)+ \\ C(s-1)(s+1)(s^2+9)+(Ds+E)(s-1)(s+1)(s+2) &= s^2+12 \end{aligned}$$

Put $s=1$:	$60A=13;$	$A=\frac{13}{60}$
Put $s=-1$:	$-20B=13;$	$B=-\frac{13}{20}$
Put $s=-2$:	$39C=16;$	$C=\frac{16}{39}$
Coefficient of s^4 :	$A+B+C+D=0;$	$D=-(A+B+C)=\frac{3}{130}$
Coefficient of s^0 :	$18A-18B-9C-2E=12;$	$E=\frac{1}{2}(18A-18B-9C-12)=-\frac{3}{65}$

Hence

$$Y(s)=\frac{13}{60}\cdot\frac{1}{s-1}-\frac{13}{20}\cdot\frac{1}{s+1}+\frac{16}{39}\cdot\frac{1}{s+2}+\frac{\frac{3}{130}s-\frac{3}{65}}{s^2+9}$$

and

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \frac{13}{60}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{13}{20}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+ \\ &\quad \frac{16}{39}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}+\frac{3}{130}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}-\frac{1}{65}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \\ &= \frac{13}{60}e^t-\frac{13}{20}e^{-t}+\frac{16}{39}e^{-2t}+\frac{3}{130}\cos 3t-\frac{1}{65}\sin 3t. \end{aligned}$$

7.

$$\mathcal{L}\{y''+y\}=\mathcal{L}\left\{\sqrt{2}\sin\sqrt{2}t\right\}$$

$$\begin{aligned} \Rightarrow \mathcal{L}\{y''\}+\mathcal{L}\{y\} &= \frac{2}{s^2+2} \\ \Rightarrow s^2Y-sy(0)-y'(0)+Y &= \frac{2}{s^2+2} \\ \Rightarrow (s^2+1)Y &= sy(0)+y'(0)+\frac{2}{s^2+2} \end{aligned}$$

$$\Rightarrow Y=y(0)\frac{s}{s^2+1}+y'(0)\frac{1}{s^2+1}+\frac{2}{(s^2+2)(s^2+1)}$$

Hence,

$$\begin{aligned} y(t) &= y(0)\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + y'(0)\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+2)(s^2+1)}\right\} \\ &= y(0)\cos t + y'(0)\sin t + 2\left(\frac{\sqrt{2}\sin t - \sin\sqrt{2}t}{\sqrt{2}(2-1)}\right) \\ &= y(0)\cos t + y'(0)\sin t + 2\sin t - \sqrt{2}\sin\sqrt{2}t \\ &= 10\cos t + 2\sin t - \sqrt{2}\sin\sqrt{2}t. \end{aligned}$$

8.

$$\mathcal{L}\{y' - y\} = \mathcal{L}\{2\cos 4t\}$$

$$\Rightarrow \ -y(0) + sY(s) - Y(s) = 2\left(\frac{s}{s^2+16}\right)$$

$$\Rightarrow \ Y(s) = 2\left(\frac{s}{(s^2+16)(s-1)}\right)$$

since $y(0) = 0$. Now,

$$\begin{aligned} \frac{s}{(s^2+16)(s-1)} &= \frac{As+B}{s^2+16} + \frac{C}{s-1} \\ &\Rightarrow \begin{cases} A = -\frac{1}{17} \\ B = \frac{16}{17} \\ C = \frac{1}{17} \end{cases} \end{aligned}$$

and so

$$Y(s) = 2\left(\left(-\frac{1}{17}\right)\left(\frac{s}{s^2+16}\right) + \left(\frac{16}{68}\right)\left(\frac{4}{s^2+16}\right) + \left(\frac{1}{17}\right)\left(\frac{1}{s-1}\right)\right).$$

Hence, the inverse Laplace transform gives

$$y(t) = -\frac{2}{17}\cos 4t + \frac{8}{17}\sin 4t + \frac{2}{17}e^t.$$

9. We have

$$\begin{aligned} \mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 1 \\ \mathcal{L}\{y'\} &= sY(s) - y(0) = sY(s) \\ \mathcal{L}\{y\} &= Y(s) \\ \mathcal{L}\{\sin 3t\} &= \frac{3}{s^2+9} \end{aligned}$$

which gives

$$2s^2Y(s) - 2 - 2sY(s) - 4Y(s) = \frac{3}{s^2+9}$$

$$\Rightarrow \quad \underbrace{\left(2s^2 - 2s - 4\right)}_{2(s+1)(s-2)} Y(s) = \frac{3}{s^2+9} + 2$$

$$\Rightarrow \quad Y(s) = \left(\frac{3}{2}\right)\left(\frac{1}{(s^2+9)(s+1)(s-2)}\right) + \frac{1}{(s+1)(s-2)}.$$

Now,

$$\begin{aligned} \frac{1}{(s^2+9)(s+1)(s-2)} &= \frac{As+B}{(s^2+9)} + \frac{C}{(s+1)} + \frac{D}{(s-2)} \\ &\Rightarrow \begin{cases} A = \frac{1}{130} \\ B = -\frac{11}{130} \\ C = -\frac{1}{30} \\ D = \frac{1}{39} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{(s+1)(s-2)} &= \frac{E}{(s+1)} + \frac{F}{(s-2)} \\ &\Rightarrow \begin{cases} E = -\frac{1}{3} \\ F = \frac{1}{3} \end{cases}. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{As+B}{(s^2+9)}\right\} &= \mathcal{L}^{-1}\left\{\frac{As}{(s^2+9)}\right\} + \mathcal{L}^{-1}\left\{\left(\frac{B}{3}\right)\frac{3}{(s^2+9)}\right\} \\ &= A\cos 3t + \frac{B}{3}\sin 3t \end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{C}{(s+1)}\right\} = Ce^{-t} \qquad \mathcal{L}^{-1}\left\{\frac{D}{(s-2)}\right\} = De^{2t}$$

$$\mathcal{L}^{-1}\left\{\frac{E}{(s+1)}\right\} = Ee^{-t} \qquad \mathcal{L}^{-1}\left\{\frac{F}{(s-2)}\right\} = Fe^{2t}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t)$$

and so

$$
\begin{aligned}
y(t) &= \frac{3A}{2}\cos 3t + \frac{B}{2}\sin 3t + \left(\frac{3C}{2} + E\right)e^{-t} + \left(\frac{3D}{2} + F\right)e^{2t} \\
&= \frac{3}{260}\cos 3t - \frac{11}{260}\sin 3t + \left(-\frac{1}{20} - \frac{1}{3}\right)e^{-t} + \left(\frac{1}{26} + \frac{1}{3}\right)e^{2t} \\
&= \frac{3}{260}\cos 3t - \frac{11}{260}\sin 3t + \left(-\frac{23}{60}\right)e^{-t} + \left(\frac{29}{78}\right)e^{2t}.
\end{aligned}
$$